Maximal ideal spaces of invariant function algebras on homogeneous spaces of compact Lie groups

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The problem

- Let G be a compact connected Lie group,
- $M = G/K$ be its homogeneous space,
- A be a G-invariant *uniform algebra* on M, i.e., a closed subalgebra of the Banach algebra $C(M)$ which contains constant functions.

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Problem

Describe maximal ideal spaces of invariant algebras.

Polynomial hulls

Let $\mathcal{P}$ be the algebra of all holomorphic polynomials on $\mathbb{C}^n$.

The polynomial hull of a compact $Q \subset \mathbb{C}^n$ is defined as follows:

$$\widehat{Q} = \{z \in \mathbb{C}^n : \, |p(z)| \leq \|p\|_Q \text{ for all } p \in \mathcal{P}\},$$

where $\|p\|_Q = \sup\{|p(q)| : \, q \in Q\}$.

The closure $P(Q)$ of $\mathcal{P}|_Q$ in $C(Q)$ is a uniform algebra on Q.

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The mapping $\varphi \to (\varphi(z_1), \dots, \varphi(z_n))$ from $P(Q)^*$ to $\mathbb{C}^n$ defines a bijection between $\mathcal{M}_{P(Q)}$ and $\widehat{Q}$.

The finite dimensional case

Let $G \subseteq \mathrm{U}(n)$ and $O_v = Gv$ be the orbit of a vector $v \in \mathbb{C}^n$.
Then $P(O_v)$ is an invariant algebra on O_v and we get a finite dimensional version of the problem:

Describe polynomial hulls of orbits of compact linear groups.

Invariant ideals

The natural representation T of G in $C(M)$ extends to the convolution algebra $M(G)$ of finite regular Borel measures on G:

$$T_\mu f(x) = \int f(g^{-1}x)\, d\mu(g),$$

where $\mu \in M(G)$, $f \in C(M)$, and $x \in M$. The operators T_μ preserve every closed invariant subspace of $C(M)$.

Let σ be the invariant probability measure of G. Then $T_\sigma f$ is constant for any $f \in C(M)$.

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Lemma

Let I be a closed invariant ideal of A such that $I \neq A$. Then $T_\sigma I = \{0\}$.

The greatest invariant ideal

Let $\mathfrak{I}$ be the set of all closed invariant ideals I in A such that $I \neq A$ ordered by inclusion.

Proposition

The family $\mathfrak{I}$ contains the greatest ideal $\mathfrak{J}$.

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Proof.

By the lemma above, for any $I \in \mathfrak{I}$ and $f \in I$

$$T_\sigma f = \int_{G\times M} f(g^{-1}x)\, d\sigma(g) dx = \int_M f(x)\, dx = 0.$$

Hence the closure $\mathfrak{J}$ of the algebraic sum of all $I \in \mathfrak{I}$ belongs to $\mathfrak{I}$. □

Two classes of invariant algebras

We define two types of invariant algebras according to the extreme cases for their greatest ideals:

$\mathfrak{A}$: $\operatorname{codim} \mathcal{J} = 1$,

$\mathfrak{B}$: $\mathcal{J} = \{0\}$.

They can be stated in terms of the natural actions of G in A and $\mathcal{M}_A$:

$$A \in \mathfrak{A} \iff G \text{ has the unique fixed point in } \mathcal{M}_A,$$
$$A \in \mathfrak{B} \iff A \text{ has no proper } G\text{-invariant ideals.}$$

Two parts of $\mathcal{M}_A$

Set $B = A/\mathcal{J}$ and $C = \mathcal{J} + \mathbb{C}$. Let ϵ denote element of $\mathcal{M}_C$ which corresponds to the maximal ideal $\mathcal{J}$. Note that ϵ is the fixed point of G in $\mathcal{M}_C$. Put

$$\mathcal{M}'_B = \{\varphi \in \mathcal{M}_A : \varphi = 0 \text{ on } \mathcal{J}\},$$
$$\mathcal{M}'_C = \{\varphi \in \mathcal{M}_A : \varphi \neq 0 \text{ on } \mathcal{J}\}.$$

Then $\mathcal{M}_A = \mathcal{M}'_B \cup \mathcal{M}'_C$ and $\mathcal{M}'_B \cap \mathcal{M}'_C = \varnothing$.

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Then $\mathcal{M}_A = \mathcal{M}'_B \cup \mathcal{M}'_C$ and $\mathcal{M}'_B \cap \mathcal{M}'_C = \varnothing$.

Let $\pi : A \to B$ be the natural projection and $\rho : A^* \to C^*$ be the operator of restriction to $\mathcal{J}$.

The mapping $\varphi \to \varphi \circ \pi$ is a homeomorphism between $\mathcal{M}_B$ and $\mathcal{M}'_B$. The restriction operator ρ is a continuous bijection between $\mathcal{M}'_C$ and $\mathcal{M}_C \setminus \{\epsilon\}$.

Here is a construction for extension of $\varphi \in \mathcal{M}_C \setminus \{\epsilon\}$ to A. It follows the proof of [1, Theorem 6.1]. Let $u \in \mathcal{J}$ be such that $\varphi(u) = 1$ and put

$$\psi(f) = \varphi(fu)$$

for $f \in A$. Then independently of the choice of u we have $\psi(1) = 1$,

$$\psi(f) = \varphi(f) \ \text{ for all } f \in \mathcal{J},$$

and

$$\begin{aligned}\psi(f_1 f_2) = \varphi(f_1 f_2 u) = \varphi(f_1 f_2 u^2) = \varphi((f_1 u)(f_2 u)) \\ = \varphi(f_1 u)\varphi(f_2 u) = \psi(f_1)\psi(f_2)\end{aligned}$$

for all $f_1, f_2 \in A$.

Thus, a possible way to solve the problem is to study algebras of the classes $\mathfrak{A}$, $\mathfrak{B}$ and method of gluing $\mathcal{M}_B$ with $\mathcal{M}_C \setminus \{\epsilon\}$.

Set

$$(a, b) = T_\sigma(ab) = \int_M a(x)b(x)\,dx.$$

This is a bilinear G-invariant form on the closure H of A in $L^2(M)$ and, moreover, $(a, \overline{b})$ is the scalar product in H.

Another characterizations of $\mathfrak{J}, \mathfrak{A}, \mathfrak{B}$

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$\mathfrak{J}$ consists of $a \in A$ such that $(a, f) = 0$ for all $f \in A$.
This is true because $(ab, f) = (a, bf) = 0$ for all $a \in \mathfrak{J}$ and $b, f \in A$.

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$A \in \mathfrak{A}$ if and only if $a, b \in A$ and $(a, 1) = 0$ imply $(a, b) = 0$.
The form has rank 1 and $\mathfrak{J} = 1^\perp$.

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$A \in \mathfrak{B}$ if and only if the form $(\ ,\ )$ is non-degenerate on A.
Due to the characterization of $\mathfrak{J}$, the form $(\ ,\ )$ degenerates on $\mathfrak{J}$.

Uniform algebras: representing measures and hulls

Recall that *a uniform algebra* on a Hausdorff compact Q is a closed subalgebra of the Banach algebra $C(Q)$ of all continuous functions on Q which contains constant functions.

A representing measure of $\varphi \in \mathcal{M}_A$ is an extension of φ onto $C(M)$ which preserves its norm. Since $\varphi(1) = 1$ and $\|\varphi\| = 1$, it is a probability measure. The set $\mathcal{M}_\varphi$ of representing measures for φ is a $*$-weakly compact convex set.

For a set $E \subseteq Q$, the *A-hull $\widehat{E}$ of E* is equal to the set of $\varphi \in \mathcal{M}_A$ such that $|\varphi(f)| \leqslant \|f\|_E$ for all $f \in A$.

Uniform algebras: representing measures and hulls

Let $\epsilon \in \mathcal{M}_A$ be the fixed point of G. Then $\mathcal{M}_\epsilon$ is G-invariant. Hence it contains the fixed point σ.

Let $E \subseteq M$ be closed, $\varphi \in \mathcal{M}_A$. Then $\varphi \in \widehat{E}$ if and only if there is a representing measure with support in E.

Antisymmetric algebras

- A uniform algebra $A \subseteq C(Q)$ is called *antisymmetric* if any real valued function in A is constant.
- A set $E \subseteq Q$ is called *a set of antisymmetry* if every real on E function $f \in A$ is constant on E.
- If two sets of antisymmetry have a common point, then their union is also a set of antisymmetry.
- Every set of antisymmetry is contained in a maximal one.

Let $\mathbb{T}$ be the unit circle $|z| = 1$ in $\mathbb{C}$ and $\mathbb{D}$ be the unit disc bounded by $\mathbb{T}$. Let $A(\mathbb{D})$ be the algebra of functions holomorphic on $\mathbb{D}$ and continuous up to the boundary. For every $f \in A(\mathbb{D})$ the Taylor series converges in $\mathbb{D}$:

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots .$$

The complex conjugate function $\overline{f}$ also admits the Taylor decomposition:

$$\overline{f(z)} = \overline{c}_0 + \overline{c}_1 \overline{z} + \overline{c}_2 \overline{z}^2 + \cdots .$$

It is not holomorphic, i.e. $\overline{f} \notin A(\mathbb{D})$. Thus $A(\mathbb{D})$ is antisymmetric.

The Bishop–Shilov decomposition

Theorem (Bishop–Shilov decomposition)

Let A be a uniform algebra on compact Q and $\mathfrak{Q}$ be the family of all maximal sets of antisymmetry for A. Then

(i) *the sets in $\mathfrak{Q}$ cover Q, they are closed and pairwise disjoint,*

(ii) *$f \in C(Q)$ belongs to A if and only if $f|_E \in A|_E$ for any $E \in \mathfrak{Q}$.*

If every set of antisymmetry is a single point, then this theorem becomes the Stone–Weierstrass theorem.

The Bishop–Shilov decomposition for invariant algebras

Let A be an invariant algebra. Denote by Q_x the set from $\mathfrak{Q}$ which contains $x \in M$ and put $o = [K]$, $A_o = A|_{Q_o}$, and $K' = \{g \in G : go \in Q_o\}$. Then

$$K' = \{g \in G : gQ_o = Q_o\}.$$

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According to the Bishop–Shilov decomposition, we have

- K' is a closed subgroup of G such that $K' \supseteq K$ and $\mathfrak{Q}$ consists of its orbits.
- A contains all K'-invariant continuous functions.
- A_o is an *antisymmetric* invariant algebra on Q_o.
- $A = \{f \in C(M) : T_g f\big|_{Q_o} \in A_o\}$.

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- A_o is an *antisymmetric* invariant algebra on Q_o.
- $A = \{f \in C(M) : T_g f\big|_{Q_o} \in A_o\}$.

The space $M = G/K$ can be treated as the total space of the equivariant fibration with the base $M' = G/K'$ and fiber $F = K'/K$. We can get $\mathcal{M}_A$ filling F with the space $\mathcal{M}_{A_o}$.

The class $\mathfrak{A}$

Let μ denote the invariant probability measure on M.

If $A \in \mathfrak{A}$, then

(i) $\mu \in \mathcal{M}_\epsilon$,

(ii) A is antisymmertric.

To prove (i), note that the convex $*$-weakly compact set $\mathcal{M}_\epsilon$ is G-invariant and μ is its fixed point.

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Let $f \in A$ be a real nonconstant function. Replacing f with $f - c$, for a suitable c we have

$$\int f \, d\mu = 0 \quad \text{and} \quad \int f^2 \, d\mu > 0.$$

On the other hand,

$$\epsilon(f^2) = \epsilon(f)^2 = \left(\int f \, d\mu \right)^2 = 0.$$

This proves (ii).

The class $\mathfrak{A}$

Recall that A_o is "the antisymmetric part" of an invariant algebra A in the Bishop–Shilov decomposition. Let $\varkappa$ be the Haar measure of K'. Then $T_\varkappa^2 = T_\varkappa$.

Theorem

If $A_o \in \mathfrak{A}$, then

- *$T_\varkappa$ is a projection onto the algebra of all K'-invariant functions in $C(M)$, which is isometrically isomorphic to $C(M')$.*
- *$T_\varkappa$ is an endomorphism of A and $\ker T_\varkappa = \mathfrak{J}$.*

The first assertion holds due to the Bishop-Shilov decomposition theorem. It is easy to prove the second.

Orbits of linear groups

Every compact group G acting in a complex linear space V admits the complexification $G^{\mathbb{C}}$. Let $v \in V$. We denote $M = O_v = Gv$, $M^{\mathbb{C}} = O_v^{\mathbb{C}} = G^{\mathbb{C}}v$ and call them *real* and *complex orbits*, respectively.

The class $\mathfrak{A}$ for orbits of linear groups

Let 0 be the unique fixed point of G in V, $M = O_v$ for some $v \in V$, and 0 belongs to the closure of $M^{\mathbb{C}}$. Then $\{0\}$ is the unique closed orbit in it. The set of all such v is the *nilpotent cone* (or *null-cone*) $\mathcal{N}$. It can be defined as follows:

$p(v) = 0$ for any G-invariant polynomial p such that $p(0) = 0$.

Set $\mathcal{P}_0 = \{p \in \mathcal{P} : \ p(0) = 0\}$. Since any invariant polynomial can be obtained by averaging over G,

$$v \in \mathcal{N} \quad \Leftrightarrow \quad \int_G p(gv)\, dg = 0 \ \text{ for all } \ p \in \mathcal{P}_0.$$

It follows that $P(M) \in \mathfrak{A}$ if and only if $v \in \mathcal{N}$. Then $\widehat{M} \subset \mathcal{N}$.

Let's denote $A = P(M)$ for brevity.

The following conditions are equivalent:

(i) $A \in \mathfrak{B}$,

(ii) $M^{\mathbb{C}}$ is closed in V.

(iii) $\widehat{M} \subset M^{\mathbb{C}}$.

(i) $\Rightarrow$ (ii): If $M^{\mathbb{C}}$ is not closed, then the ideal generating by polynomials which vanish on the closed orbit in the closure of $M^{\mathbb{C}}$ is invariant and proper in A.
(ii) $\Rightarrow$ (iii): If $p \in \mathcal{P}$ vanishes on M, then $p = 0$ on $\widehat{M}$ by definition of the polynomial hull. Hence points of $\widehat{M}$ satisfy the finite number of equations which distinguish $M^{\mathbb{C}}$.
(iii) $\Rightarrow$ (i): Let I be a maximal ideal which contains $\mathfrak{J}$. Since $\widehat{M} \subset M^{\mathbb{C}}$, there is $u \in M^{\mathbb{C}}$ such that $I = \{f \in A :\ f(u) = 0\}$. If $f \in \mathfrak{J}$, then $f = 0$ on O_u. Hence $f = 0$ on its complexification $M^{\mathbb{C}}$. It follows that $\mathfrak{J} = 0$.

Commutative homogeneous spaces

Here are three equivalent definitions of a commutative homogeneous space (G is compact and connected).

(1) M is *multiplicity free:* the quasiregular representation of G in $L^2(M)$ contains every irreducible unitary representation of G with multiplicity 0 or 1.

(2) (G, K) is a *Gelfand pair:* the convolution algebra of all left and right K-invariant functions in $L^1(G)$ (or measures) is commutative.

(3) $M^{\mathbb{C}}$ is *spherical:* a Borel subgroup of $G^{\mathbb{C}}$ has an open orbit in $M^{\mathbb{C}}$.

Theorem (G.,[3], 2008)

Let A be an invariant algebra on a commutative homogeneous space M. Then

- *if A is antisymmetric, then $A \in \mathfrak{A}$,*
- *if $A \in \mathfrak{B}$, then $\overline{A} = A$.*

The assumption of commutativity is essential. There are antisymmetric $A \in \mathfrak{B}$.

Corollary

Let M be commutative. Then the projection $A \to B = A/\mathfrak{J}$ extends to an endomorphisms of A with kernel $\mathfrak{J}$. The algebra B is isometrically isomorphic to $C(M')$ for some homogeneous space M'.

Commutative homogeneous spaces as real forms

A real form of a complex manifold is the set of fixed point of its antiholomorphic involution. In what follows the real forms are compact.

Corollary

Let $M^{\mathbb{C}}$ be closed. If a real orbit M in $M^{\mathbb{C}}$ is a commutative homogeneous space, then M is a real form of $M^{\mathbb{C}}$.

Real forms of closed complex orbits can be characterized by their polynomial convexity.

Theorem (G., Latypov, [4], 2001)

Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a compact connected group, $v \in \mathbb{C}^n$. Then

(i) *$M^{\mathbb{C}}$ is closed if and only if it contains a polynomially convex real orbit,*

(ii) *if $M^{\mathbb{C}}$ is closed, then $\widehat{M} = M$ if and only if M is a real form of $M^{\mathbb{C}}$,*

(iii) *if $M^{\mathbb{C}}$ is closed and M is a real form of $M^{\mathbb{C}}$, then $P(M) = C(M)$. The converse is also true.*

Example of an antisymmetric algebra from $\mathfrak{B}$

Set $G = \mathrm{SU}(2)$. It is a real form of the group $G^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ defined by the antiholomophic involution $X \to (\overline{X}^{\mathsf{T}})^{-1}$. Let's consider the adjoint action of G in the Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathrm{sl}(2, \mathbb{C})$: $\mathrm{Ad}(g)\zeta = g^{-1}\zeta g$. Here is the standard basis of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathsf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathsf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The group G consists of matrices $\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix}$, where $|a|^2 + |b|^2 = 1$.

Actually, in $\mathrm{su}(2) \cong \mathrm{so}(3)$ acts the group $\mathrm{SO}(3)$ whose complexification is $\mathrm{SO}(3, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$.

Example of an antisymmetric algebra from $\mathfrak{B}$

Let $\langle \zeta, \omega \rangle$ be the bilinear form $\operatorname{Tr} \zeta\omega$ on $\mathrm{su}(2)$. Set

$$v = \mathsf{h} + r\mathsf{e} = \begin{pmatrix} 1 & r \\ 0 & -1 \end{pmatrix},$$

where $r > 0$. Then $M^{\mathbb{C}}$ is the quadric $\langle \zeta, \zeta \rangle = 2$ and

$$M = O_v = M^{\mathbb{C}} \cap \{\langle \zeta^*, \zeta \rangle = 2 + r^2\} \cong \mathbb{RP}^3 \cong \mathrm{SO}(3),$$

where $\zeta^* = -\bar{\zeta}^{\mathsf{T}}$.

Setting $r = 0$ we get the real form $M^{\mathbb{R}} \cong \mathbb{S}^2$ of $M^{\mathbb{C}}$.

Example of an antisymmetric algebra from $\mathfrak{B}$

Note that $M^{\mathbb{C}}$ contains the line $L = \mathsf{h} + \mathbb{C}\mathsf{e}$. Clearly,

$$\mathsf{h} + \mathbb{T}\rho\mathsf{e} \subseteq M,$$
$$\mathsf{h} + \mathbb{D}\rho\mathsf{e} \subseteq \widehat{M},$$

where $\mathbb{T}$ is the unit circle that bounds the unit disc $\mathbb{D}$, and $\rho = \sqrt{2 + r^2}$. It follows that

- we get $\widehat{M}$ applying $\mathrm{Ad}(g)$ to the disc $\widehat{M} \cap L$, where g runs over $\mathrm{SU}(2)$,
- this also proves that $P(M)$ is antisymmetric,
- it belongs to $\mathfrak{B}$ because $M^{\mathbb{C}}$ is closed.

Example of an antisymmetric algebra from $\mathfrak{B}$

The algebra $A = P(M)$ can be defined as the algebra of all continuous functions on M which admit holomorphic extension to the disc in the line $gL = g(\mathsf{h} + \mathbb{C}\mathsf{e})$ bounded by the circle $M \cap gL$ for all $g \in \mathrm{SU}(2)$.

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There is another definition of A. Since $\dim_{\mathbb{R}} M = 3$ and $\dim_{\mathbb{C}} M^{\mathbb{C}} = 2$, the tangent space $T_p M$ contains a complex line for all $p \in M$. This is a CR-structure on M. It defines the algebra of smooth CR-functions whose differentials are complex linear on on these lines. The uniform closure of this algebra is equal to the algebra A.

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Problem

Describe maximal ideal spaces of antisymmetric algebras without proper invariant ideals.

Remark

A *CR*-manifold is called *embeddable* if it admits an embedding into $\mathbb{C}^n$ which induces its *CR*-structure. The *CR*-structure of the example above on $M \cong \mathrm{SO}(3) \cong S^3/\mathbb{Z}_2$ can be lifted to S^3. It is known as an example of *a locally embeddable CR-manifold which is not embeddable globally*.

Let $K, K_{\mathbb{C}}$ be the stable subgroups of v in G and $G^{\mathbb{C}}$, respectively. Clearly, $K^{\mathbb{C}} \subseteq K_{\mathbb{C}}$. If $K_{\mathbb{C}} = K^{\mathbb{C}}$, then M is a real form of $M^{\mathbb{C}}$. Every $h \in K_{\mathbb{C}} \setminus K^{\mathbb{C}}$ can be associated with an analytic annulus or strip attached to M.

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There is the polar decomposition

$$h = gs,$$

where $g \in G$, $s = \exp(i\xi)$, and $\xi \in \mathfrak{g}$. Since $hv = v$, we have

$$sv = g^{-1}v \in M.$$

Setting

$$\lambda(z) = \exp(z\xi)v, \quad 0 < \mathrm{Im} z < 1,$$

we get an analytic strip attached to M.

Let X, Y be compact. We consider $C(X \times Y)$ as a completion of $C(X) \otimes C(Y)$ and denote it as $C(X) \widehat{\otimes} C(Y)$. It is a bialgebra with the comultiplication $\Delta f(g, h) = f(gh)$.

Theorem

A closed linear subspace $A \subseteq C(G)$ is an bi-invariant algebra if and only if it satisfies the following conditions:

$$1 \in A,$$
$$A \cdot A \subseteq A,$$
$$\Delta A \subseteq A \widehat{\otimes} A.$$

Bi-invariant algebras as sub-bialgebras of $C(G)$

For any $f \in A$ the function $f(gh)$ can be approximated uniformly by sums $\sum u_j(g)v_j(h)$, where $u_j, v_j \in A$. For every couple $\mu, \nu \in M(G)$

$$\mu \otimes \nu(u \otimes v) = \mu(u_j)\nu(v_j).$$

If $\mu \in A^\perp$ or $\nu \in A^\perp$, then $\mu(u)\nu(v) = 0$. Hence $A^\perp$ is an ideal of $M(G)$ and $A^\perp$ is $G \times G$-invariant.

Bi-invariant algebras on groups

For any compact group G, the space $G \times G/G \cong G$ is commutative. If A is an $G \times G$-invariant algebra on G, then

- there is a natural semigroup structure in $\mathcal{M}_A$,
- the inversion in G extends to an involutive antiholomorphic antiautomorphism $^* : \mathcal{M}_A \to \mathcal{M}_A$,
- each $\varphi \in \mathcal{M}_A$ admits the polar decomposition $\varphi = gs$, there $g \in G$ and s is symmetric (i.e., $s^* = s$),

There is a natural preorder in the set $\mathcal{I}$ of idempotents:

$$j \preceq k \;\Leftrightarrow\; jk = kj = j.$$

- The set $\mathcal{I}$ is a complete lattice.

In particular, there is the least idempotent ϵ in $\mathcal{I}$.

- For any symmetric s, there exists the unique one parameter symmetric semigroup $\gamma : \mathbb{R}^+ \to \mathcal{M}_A$ such that $\gamma(1) = s$,
- it extends to a homomorphism $\mathbb{C}^+ \to \mathcal{M}_A$ such that $f \circ \gamma$ is holomorphic on $\mathbb{C}^+$,
- it has a limit $\lim_{\mathrm{Re}\, z \to \infty} \gamma(z)$ in $\mathcal{I}$.

Every $j \in \mathcal{I}$ relates to a subgroup G^j of $\mathcal{M}_A$ with the unit j. Moreover, there is a convex closed pointed $\mathrm{Ad}(G)$-invariant cone C^j in the Lie algebra $\mathfrak{g}^j$ such that the set $G^j \exp(iC^j)$ is a subsemigroup of the complexification of G^j. It is a complex Lie semigroup. The family of these semigroups covers $\mathcal{M}_A$.

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Thank you for your attention!