Burnside type Problems in the theory of groups and loops.

Alexandr Grishkov

University of São Paulo, Bazil

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Classical Burnside Problems in the Theory of Groups

General Burnside Problem - GPB

Is it true that a finitely generated periodic group is finite?

Positive solutions of General Burnside Problem are known for some classes:

W.Burnside (1905) I.Shur (1911) - linear periodic groups

O.Smidt (1945) - solvable periodic groups.

E. Golod and I. Shafarevich (1964) constructed a series of finitely generated infinite p-groups for an arbitrary prime p.

The (ordinary) Burnside Problem - OBP

Is it true that a finitely generated group of exponent n is finite?
There were obtained positive solutions in these particular cases:
W.Burnside (1902) solved for $n = 3$
I.N.Sanov (1940) solved for $n = 4$
M.Hall (1957-1958) solved for $n = 6$
Till now there are not known positive solutions for $n = 5, n > 6$.
P.Novikov and S. Adian (1968) constructed a finitely generated infinite group of odd exponent $n \geq 4381$. Later S.I. Adian (1975) improved this estimation to odd exponent $n \geq 665$ and S. V. Ivanov (1994) gave the negative solution for sufficiently large even exponents divisible by a large power of 2.
In 1982, A. Yu. Ol'shanskii gave counterexamples for odd exponents (greater than 10^{10}), using geometric ideas which provided a considerably simpler proof.

Restricted Burnside Problem- RBP
(formulated in the 1930s)
Is it true that for each m, n there are only finitely many finite $n-$generated groups of exponent m?
In the case of the prime exponent p, this problem was extensively studied by A. I. Kostrikin during the 1950s, prior to the negative solution of the general Burnside problem. He proved (using a relation with deep questions about identities in Lie algebras in finite characteristic) the existence of the biggest finite group $B_0(m, p)$ of exponent p with m generators.
Using Hall-Higman Reduction Theorem (1956) and the classification of simple groups the case of arbitrary exponent has been completely settled in the affirmative by Efim Zelmanov, who was awarded the Fields Medal in 1994 for his work.

Burnside type problems allow a natural generalization to the case when we have not information about the exponent of the whole group, but about some part of it. The most interesting situation is when the group G with some involution σ, generated by the set $G_\sigma = \{x^{-1}x^\sigma | x \in G\}$. Suppose that $x^n = 1, x \in G_\sigma$, what can be said about the group G? Is there a finite number of such finite groups with m generators? In the general case, the answer is negative even for $n = 2$. But there is one important case when the answer is *yes*. Suppose that the group G additionally admits a third-order automorphism of ρ and $< \sigma, \rho | \rho^\sigma = \rho^{-1} > \simeq S_3$. Moreover for every $a \in G_\sigma$ we have $aa^\rho a^{\rho^2} = 1$. Such groups are called groups with triality. Our main result can be stated:

Theorem (G., Sabinina, Zelmanov)
arxiv.org/pdf/2005.11824.pdf. There is a finite number of groups with triality G with m generators such that $x^n = 1, x \in G_\sigma$, $n = p^k$, $p-$ prime > 3.

Loops and Groups

Let L be a set with a binary operation $\cdot$ with unit e:
$e \cdot x = x \cdot e = x \in L$.
$(L, \cdot)$ is a **loop** if the mappings $L_a : x \to a \cdot x, R_b : x \to x \cdot b$ are bijections, $\forall a, b \in L$

multiplication group $Mult(L)$ - group generated by $\{L_a, R_b\}_{a,b \in L}$
inner mapping group $Int(L) = \{\phi \in Mult(L) \mid \phi(e) = e\}$

A **normal subloop** is the kernel of loop homomorphism. A subloop is normal if and only if it is invariant under inner mappings.

In some sense we can consider loops as factors of groups by non-normal subgroups. Let G be a group and H be a subgroup of G. Let fixed a set $B \subset G$ such that $G = HB$ and every $g \in G$ has unique form $g = hb$, $b \in B, h \in H$. Then the set B admits a binary product $b_1.b_2 = b_3$, where $b_1 b_2 = hb_3$. Usually this product is not associative. In some cases $(B, .)$ is a loop.

A notion of group with triality was introduced by Glauberman and Doro and has intimate relation with *Moufang loop*.
A loop U is called a Moufang loop if it satisfies the following identity:

$$(x(yz))x = (xy)(zx)$$

If G is a group with triality and $G_\sigma = \{x^{-1}x^\sigma | x \in G\}$ then G_σ is a Moufang loops with operation $a.b = a^{-\rho}ba^{-\rho^2} \in G_\sigma$ and every Moufang loop may be obtained in this way. Our main Theorem in **arxiv.org/pdf/2005.11824.pdf** was formulated in the following form:
Theorem (G., Sabinina, Zelmanov). There is a finite number of Moufang loops M with m generators such that $x^n = 1, x \in M$, $n = p^k$, $p-$ prime > 3.

For $n = p$ this theorem was proved in 1987 by A. Grishkov (if $p \neq 3$)[1] and G. Nagy (if $p = 3$)[2].
It is analogue of Theorem of A.I.Kostrikin for groups.

[1] A.N. Grishkov, *The weakend Burnside problem for Moufang loops of prime period,* Sibirsk.Mat.Zh. 28 (1987), no. 3, 60-65.
[2] G. P. Nagy, *Burnside problems for Moufang and Bol loops of small exponent,* Acta Sci Math. (Szeged) 67 (2001), no. 3-4, 687-696.

G.Nagy proved that finitely generated Moufang loop of exponent 3 is finite [2]. He reduced the proof to the commutative case where it is easy corollary of the famous Bruck-Slaby Theorem on nilpotency of commutative Moufang loops.
It is unique known positive non-trivial solution of *OBP* for Moufang loops. We have one trivial case $p = 2$, since Moufang loops of exponent 2 are abelian groups. We hope that all finitely generated Moufang loops of exponent 4 (analog of Sanov Theorem) and 6 (analog of Hall Theorem) are finite.
The most known examples of Moufang loops are the loops of units in octonion rings $\mathbf{O}(\mathbf{R})$, $R-$commutative associative ring.

Lie and Malcev algebras

Let $\mathbf{F}_p$ be a field of order p, let G be a group. Consider the group algebra $\mathbf{F}_p G$ and its fundamental ideal ω, spanned by all elements $1-g$, $g \in G$. The Zassenhaus filtration is the descending chain of subgroups

$$G = G_1 > G_2 > \cdots$$

where $G_i = \{g \in G | 1-g \in \omega^i\}$. Then $[G_i, G_j] \subseteq G_{i+j}$ and each factor G_i/G_{i+1} is an elementary abelian p-group. Hence,

$$L = L_p(G) = \sum_{i \geq 1} L_i, \quad L_i = G_i/G_{i+1}$$

is a vector space over $\mathbf{F}_p$.

The bracket

$$[x_i G_{i+1}, y_j G_{j+1}] = [x_i, y_j] G_{i+j+1}; \;\; x_i \in G_i, y_j \in G_j,$$

makes L a Lie algebra. Notice that the bracket [,] on the left hand side of the last equality is a Lie bracket whereas [,] on the right hand side denotes the group commutator.

We call a Lie algebra (resp. Lie p-algebra) L with automorphisms ρ, σ a Lie algebra with triality if $\rho^3 = \sigma^2 = (\rho\sigma)^2 = 1$ and for an arbitrary element $x \in L$ we have

$$(x^\sigma - x) + (x^\sigma - x)^\rho + (x^\sigma - x)^{\rho^2} = 0.$$

Lemma

Let G be a group with triality and let p be a prime number. Then $L_p(G)$ is a Lie p-algebra with triality.

Recall that a (nonassociative) algebra is called a Malcev algebra if it satisfies the identities

1. $xy = -yx$
2. $(xy)(xz) = ((xy)z)x + ((yz)x)x + ((zx)x)y$.

Lemma

Let L be a Lie algebra with triality over a field of characteristic $\neq 2, 3$. Let $H = \{x \in L | x^\sigma = -x\}$. Then H is a Malcev algebra with multiplication

$$a * b = [a + 2a^\rho, b] = [a^\alpha, b],$$

where $a, b \in H$, $\alpha = 1 + 2\rho$.

Lemma

For arbitrary elements $a, b, c \in H$ we have

$$3[[a,b],c] = 2(a*b)*c + (c*b)*a + (a*c)*b.$$

We remark that in a Lie algebra with triality over a field F, for arbitrary elements $a_1, \dots, a_n \in H$ the subspace $\sum_{i=1}^{n} Fa_i + \sum_{i=1}^{n} Fa_i^{\alpha} = \sum_{i=1}^{n} Fa_i + \sum_{i=1}^{n} Fa_i^{\rho}$ is invariant with respect to the group of automorphisms $\langle \sigma, \rho \rangle$.

Lemma

If a Lie algebra L with triality is generated by elements $a_1, \dots, a_m, a_1^{\alpha}, \dots, a_m^{\alpha}$, where $a_1, \dots, a_m \in H$, then the Malcev algebra H is generated by $a_1, \dots, a_m$.

Consider again a group with triality G and the Lie algebra with triality $L = L_p(G) = \sum_{i=1}^{\infty} L_i$. The subspace $H = \{a \in L | a^\sigma + a = 0\}$ is graded, i.e. $H = \sum_{i=1}^{\infty} H_i$, $H_i = H \cap L_i$.

Lemma

Suppose that for an arbitrary element $g \in G$ we have $[g, \sigma]^{p^n} = 1$. Then

1. *for an arbitrary homogeneous element $a \in H_i, i \geq 1$, we have $ad(a)^{p^n} = 0$,*
2. *for arbitrary homogeneous elements $a_1, \ldots, a_{p^n}$ from H we have*

$$\sum_{\pi \in S_{p^n}} ad(a_{\pi(1)}) \cdots ad(a_{\pi(p^n)}) = 0.$$

Local nilpotence in Malcev algebras

Proposition

Let $M = M_1 + M_2 + \cdots$ be a finitely generated graded Malcev algebra over a field of characteristic $p \neq 2, 3$, such that

1. $ad^*(a)^{p^n} = 0$ *for an arbitrary homogeneous element* $a \in M$,
2. $\sum_{\pi \in S_{p^n}} ad^*(a_{\pi(1)}) \cdots ad^*(a_{\pi(p^n)}) = 0$ *for arbitrary* $a_1, \ldots, a_{p^n} \in M$. *Here* $ad^*(a) : x \to x * a$.

Then the Malcev algebra M is nilpotent and finite dimensional.

Lemma

(V.T. Filippov) A finitely generated solvable Malcev algebra over a field of characteristic > 3 is nilpotent if and only if each of its Lie homomorphic images is nilpotent.

The other periodic loops.

Bol loops.

Recal that a loop $(B, *)$ is a BOL loop if $x * ((y * z) * y) = ((x * y) * z) * y$ for all x, y, z from B. Every Bol loop may be realized as "semisubgroup"of some group $G = BH$, $B \cap H = 1$, it means that $aba \in B$, if $a, b \in B$.

Theorem. Free two generated Bol loop of exponent two is infinite.

Definition. Let B be a Bol loop of exponent two. A set $T \subset B$ is a *basis* of B if

(i) the set T is a prebasis, it means that every $b \in B$ has a form $b = b_1 b_2 .... b_{n-1} b_n b_{n-1} ... b_2 b_1,\ b_i \in T$;

(ii) the set T is "independent", it means the the form $b = b_1 b_2 .... b_{n-1} b_n b_{n-1} ... b_2 b_1,\ b_i \in T$; is unique.

Theorem(G., Rasskazova, G.Souza)

Every free Bol loop of exponent two has a basis.
Let B_n be a free two generated Bol loop of exponent n.
It is clear that B_n is infinite if n is even.
The following problem, formulated by G.Nagy is very impotent and difficult: is the loop B_3 finite?
We think that B_3 is infinite but the corresponding group of right multiplications
$Mult_r(B_3) =< R_x | x \in B_3, R_x : B_3 \to B_3, R_x(b) = b * x >$
is almost locally finite: for every finite subset $S \subset B_3$ the corresponding group $Mult_S =< R_x | x \in S >$ is finite.
We note that in the case of Bol loops of exponent two it is not true.

Theorem(G.+Anderson+Kinyon+G.Nagy)

(i) Free Bol loop of exponent three nilpotent of class two is a Moufang loop: $(xy)(zx) = x(yz)x)$.
(ii) Free Bol loop of exponent three nilpotent of class three is not Moufang and has order 81

Theorem (Grishkov+Rasskazova+C.Sporh)

Free Bol loop of exponent two nilpotent of class two with n generators has the order 2^{b_n}; where
$b_n = 2^n(n-1) - n(n-1)/2 + 1.$

Theorem (G.+R.Miguel Pires)

Let M_n be a free Moufang loop nilpotent of class two exponent four, where $C(M_n)$ and $M_n/C(M_n)$ are elementary abelian $2-$groups.
Then M_n has order 2^m, $m = 2n + n(n-1)/2 + n(n-1)(n-2)/6$.
Moreover, if $M_n = V \oplus \bar{V} \oplus (V \wedge V) \oplus (V \wedge V \wedge V)$,
$V = \mathbf{F}_2^n = \{\sigma | \sigma \subseteq I_n = (1, ..., n)\}$, then
$\sigma.\mu = \sigma \Delta \mu + \prod_{i \in \sigma \cap \mu} \bar{i} + \prod_{i > j \in \mu, i \in \sigma} i \wedge j + \prod_{i \in \sigma, j,k \in \mu, k > j} i \wedge j \wedge k.$

Open problem

Describe the free diassociative loop D_n with n generators nilpotent of class two such that $C(D_n)$ and $D_n/C(D_n)$ are elementary abelian $2-$groups.

Theorem (G.+Rasskazova+Stulh) Let $N < D_n$ be a normal subloop of D_n generated by $\{x^2 | x \in D_n\}$.

Then $|D_n/N| = 2^s$, $s = \frac{1}{3}(2^{2n}+1) - 3.2^{n-1} + n + 1$.

THANKS!!!