Identities of Spaces of Linear Transformations and Nonassociative Linear Algebras

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Basic Definitions

Let F be a field, A be a linear associative F-algebra and E is a subspace in A (but E not necessary subalgebra of A) which generate A how linear F-algebra.

In this case, we call E a *multiplicative vector space* (in short, an *L-space*) over the field F. The algebra A will be called *enveloping for the space E*, and the space E will be called *embedded in the algebra A*.

The identity of an L-space E over a field F (embedded in an F-algebra A) is an associative polynomial $f(x_1, x_2, \ldots, x_n)$ which equal to zero in A if, instead of its variables $x_1, x_2, \ldots, x_n$ we substitute any elements from E.

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In this form, the concept of a multiplicative vector space and his identity was introduced in 2010 by I. M. Isaev and the speaker. However, (direct) analogs of this concept have been studied earlier.

In 1978 I. V. L'vov considered algebras of the form $\overline{V} = V \oplus E$, where V is a vector space and $E \subseteq \mathrm{End}_F V$. Nonzero products of elements of this algebra are given by the rule: $v_i e_{ij} = v_j$. It's clear that $\overline{V} \in \mathrm{Var}\langle x(yz) = 0\rangle$. The nonassociative polynomial $zf(R_{x_1}, R_{x_2}, \ldots, R_{x_n})$ is an identity of the algebra $\overline{V}$ iff the associative polynomial $f(x_1, x_2, \ldots, x_n)$ is equal to zero when substituting instead of variables linear combinations of elements from E.

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In 1973, Yu. P. Razmyslov introduced the concept of a weak identity of an associative Lie pair (A, L), where L is a Lie algebra and A is its associative enveloping.

A *weak identity* of a pair (A, L) is an associative polynomial $f(x_1, x_2, \ldots, x_n)$ that is equal to zero in the algebra A when substituted instead of variable $x_1, x_2, \ldots, x_n$ elements of the algebra L.

Following the above construction, the identity of the multiplicative vector space E (with the enveloping algebra A) can be considered (if necessary) as a weak identity of the pair (A, E). The pair (A, E) in this case will be called a *multiplicative vector pair*.

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Consider the set

$$E_0 = \left\{ \begin{pmatrix} \alpha & \alpha \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in F \right\}.$$

This set is a vector space over the field F, but it is not an F-algebra. The algebra $T_2(F)$ of upper triangular matrices is the enveloping algebra for E_0.

Consider the polynomial $\mathrm{St}_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$.

Since $\dim_F E_0 = 2$, the polynomial $\mathrm{St}_3(x_1, x_2, x_3)$ is the identity of the F-space E_0.

But this polynomial is not an identity of the algebra $T_2(F)$, because $\mathrm{St}_3(e_{11}, e_{12}, e_{22}) = e_{12} \neq 0$.

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Identities of Vector Spaces

Let $F\langle X\rangle$ be a free associative algebra and $\varnothing \neq G \subseteq F\langle X\rangle$. By $T(G)$ we denote the T-ideal of the algebra $F\langle X\rangle$ generated by the set G, and by $L(G)$ we denote the ideal of $F\langle X\rangle$ generated by the polynomials of the set G (as an ideal) and closed with respect to substitutions instead of variables of linear combinations of variables. These ideals will be called *L-ideals*. It's clear that $L(G) \subseteq T(G)$.

The set of all identities of a vector space E is a L-ideal of $F\langle X\rangle$. We will denote such a L-ideal by $L(E)$.

The converse is also true: every L-ideal of the algebra $F\langle X\rangle$ is the set of identities of some L-space.

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The identity f of the L-space E follows from the identities $f_1, f_2, \dots$ if $f \in L(f_1, f_2, \dots)$.

Thus, for obtaining a corollary from the identity $f(x_1, x_2, \dots, x_n)$, instead of variables $x_1, x_2, \dots, x_n$, only linear combinations of variables can be substituted. If we substitute the product of variables instead of variables $x_1, x_2, \dots, x_n$ in $f(x_1, x_2, \dots, x_n)$, the resulting polynomial may not be an identity of the L-space.

For example, the space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ satisfies the identity $\mathrm{St}_3(x, y, z) = 0$. However, E_0 does not satisfy the identity $\mathrm{St}_3(xt, y, z) = 0$. Indeed, if $x = e_{11} + e_{12}$, $t = e_{22}$, $y = e_{11} + e_{12}$, $z = e_{22}$ then $\mathrm{St}_3(xt, y, z) = \mathrm{St}_3(e_{12}, e_{11} + e_{12}, e_{22}) = -e_{12} \neq 0$.

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If there exists a finite set G of identities of a multiplicative vector space E, from which all the identities of this space follow, then the space E is called a *finitely based L-space* (*FB-space*) with a *basis of identities* G.

If such a finite set does not exist for the L-space E, then we say that the L-space E is *infinitely based* or *not finitely based* (*NFB-space*).

FB-algebras and NFB-algebras are similarly defined in other classes of algebras which may be related to Specht's problem [Specht, 1950].

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The construction of examples of NFB-algebras is an important direction in the study of varieties of algebras.

There are known examples of NFB-algebras in various classes of algebraic systems:

- in the class of groupoids [Lyndon, 1954];
- in the class of semigroups [Perkins, 1969];
- in the class of rings and linear algebras [Polin, 1976];
- in the class of loops [Vaughan-Lee, 1979].

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Proposition 1.1 [Isaev, K., 2010].

Let F be an infinite field of arbitrary characteristic. The L-spaces $A_1 = \langle e_{11}, e_{12}\rangle_F$ and $A_2 = \langle e_{11}, e_{21}\rangle_F$ over the field F is an FB-spaces with a bases of identities $\{[x, y]z\}$ and $\{x[y, z]\}$ respectively.

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$$\{\mathrm{St}_3(x, y, z), x[y, u]v, [x, y][u, v], [x, y]z_1 z_2 \ldots z_m[u, v] | m = 1, 2, \ldots\}.$$

Note that the space $A = A_1 \oplus A_2$ is an NFB-space over an arbitrary infinite field. However, any linear associative algebra over a field of characteristic zero has a finite basis of its identities [Kemer, 1987].

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Let $F = GF(q)$ be an finite field of q elements. The L-spaces $A_1 = \langle e_{11}, e_{12}\rangle_F$ and $A_2 = \langle e_{11}, e_{21}\rangle_F$ over the field F is an FB-spaces with a bases of identities $\{[x, y]z, (x^q - x)y\}$ and $\{x[y, z], x(y^q - y)\}$ respectively.

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Theorem 1.3 [Isaev, K., 2010].

Let F be an infinite field of arbitrary characteristic. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an NFB-space with a basis of identities:

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Theorem 1.5 [Isaev, K., 2011].

Let F be a field of characteristic zero. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-space with a basis of identities:

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Let $F = GF(q)$. The multiplicative vector space $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$ over the field F is an NFB-space with a basis of identities:

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A finite NFB-algebra (of arbitrary signature) is called *inherently nonfinitely based* (*INFB-algebra*) if any locally finite variety containing this algebra does not have a finite basis of identities. Sometimes inherently nonfinitely based algebras are called *essentially nonfinitely based.*

Any finite algebra containing an INFB-algebra as a subalgebra has no finite basis of identities. Thus, if we construct an example of INFB-algebra, then a series of examples of NFB-algebras is automatically constructed.

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- in the class of semigroups [Sapir, 1987];
- in the class of rings and linear algebras [I., 1989].

In 1987 M. V. Sapir gave a complete description of INFB-semigroups.

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Let $F = GF(q)$ be an finite field of q elements. The multiplicative vector space $T_2(F)$ of upper triangular matrices over the field F is an INFB-space.

It follows from this theorem that the multiplicative vector space of matrices of any order over a finite field has no finite basis of identities.

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Consider a monomial $w = w(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_k) \in F\langle X\rangle$ that is linear in each of the variables $x_1, x_2, \ldots, x_n$. Let $C_n^{(w)} = 0$ be the Capelli identity and $\mathrm{Cap}(n) = \mathrm{Var}\langle C_n^{(w)} = 0\rangle$ the variety of linear algebras satisfying all possible Capelli identities for a fixed n.

A variety $\mathfrak{M}$ of linear algebras over a field F is called *strongly nonfinitely based* (*SNFB-variety*) if $\mathfrak{M} \subseteq \mathrm{Cap}(k)$ for some k and any variety of F-algebras containing $\mathfrak{M}$ and contained in $\mathrm{Cap}(n)$ for some n is NFB-variety.

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Theorem 3.1 [K., 2015].

Let F be a field of characteristic zero, A a finite dimensional L-space over the field F, which is also an F-algebra with a unity element. An L-space A has a finite basis of identities iff $T_2(F) \notin \mathrm{Var}_L A$.

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Let F be an arbitrary field. An infinitely based L-variety $\mathcal{M}$ is the union of the Specht L-varieties $\mathcal{M}_1$ and $\mathcal{M}_2$.

Corollary 3.1.

Let F be an infinite field. An arbitrary L-space over the field F satisfying either the identity $[x, y]z = 0$ or the identity $x[y, z] = 0$ has a finite basis of identities.

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Let $F = GF(q)$. An arbitrary L-space over the field F satisfying either the identities $[x, y]z = 0$ and $(x^q - x)y = 0$ or the identities $x[y, z] = 0$ and $x(y^q - y) = 0$ has a finite basis of identities.

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The following result of V. S. Drensky strengthens corollary 3.1 and corollary 3.2.

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Over an arbitrary field every L-ideal which contains one of the weak polynomial identities $[x, y]z$ or $x[y, z]$ is finitely generated.

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Theorem 3.5 [Drensky, 2021?].

Over a field of characteristic zero the following L-ideals are all L-ideals which contain the weak identity $[x, y]z$:

- The L-ideal generated by $[x, y]z$;
- The L-ideal generated by $[x, y]z$ and the weak identity $x^n[x, y]$, $n \geq 0$;
- The L-ideal generated by $[x, y]z$ and the weak identity x^m, $m \geq 1$;
- The L-ideal generated by $[x, y]z$ and the weak identities x^m, $m \geq 2$ and $x^n[x, y]$, $0 \leq n \leq m - 2$.

There is also a dual theorem for the weak identity $x[y, z]$.

Let further F be a field of characteristic zero and $\mathcal{N}_1 = \mathrm{Var}_L\langle xy[z,t] = 0\rangle$, $\mathcal{N}_2 = \mathrm{Var}_L\langle [x,y]zt = 0\rangle$.

Theorem 3.6 [K., Unpublished].

Let F be a field of characteristic zero. L-varieties $\mathcal{N}_1$ and $\mathcal{N}_2$ are Specht.

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Recall that a T*-ideal generated by the set* G (in the notation $T(G)$) is an ideal of the algebra $F\langle X\rangle$ such that $f(w_1, w_2, \ldots, w_n) \in F\langle X\rangle$ for all $f(x_1, x_2, \ldots, x_n) \in T(G)$, $w_1, w_2, \ldots, w_n \in F\langle X\rangle$ and $T(G)$ is the smallest ideal containing G. The set of all identities in linear algebra forms a T-ideal.

An ideal of the algebra $F\langle X\rangle$ that is closed under linear permutations of variables will be called an L*-ideal*. The smallest L-ideal containing the set G will be called the L*-ideal generated by the set* G. It is clear that the set of identities of a multiplicative vector space forms an L-ideal.

Proposition 3.1 [K., 2018, 2022].

Let be $G \subseteq F\langle X\rangle$. If $[x, y]z \in G$ ($x[y, z] \in G$) then $L(G) = T(G)$. However, If $[x, y]zt \in G$ ($xy[z, t] \in G$) then $L(G) \subsetneq T(G)$

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Let $M_2(F)$ be the algebra of second-order matrices over a finite field $F = GF(q)$. Pair $(M_2(F), M_2(F))$ is critical.

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Let $T_2(F)$ be the algebra of upper triangular matrices of second-order over a finite field $F = GF(q)$. Pair $(T_2(F), T_2(F))$ is critical.

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Let $F = GF(q)$, $A_1 = \langle e_{11}, e_{12}\rangle_F$, $A_2 = \langle e_{11}, e_{21}\rangle_F$. Pairs (A_1, A_1) and (A_2, A_2) are critical.

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A variety $\mathfrak{M}$ of linear algebras is called a *almost θ-variety* if $\mathfrak{M}$ does not satisfy the θ property, but any proper subvariety of the variety $\mathfrak{M}$ satisfies this property.

Almost θ-varieties of algebras play an important role in the study of varieties of linear algebras.

In the theory of associative rings and linear algebras, almost θ-varieties are related to the indicator characterization of varieties.

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- almost commutative varieties of rings [Maltsev, 1976];
- almost commutative varieties of Φ-algebras, where Φ is a Noetherian commutative Jacobson ring with unity element [Maltsev, 1996];
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Description problems can be formulated for almost θ-varieties of L-spaces.

Theorem 3.10 [K., 2018].

Let F be a finite field and $\mathcal{M}$ a nonnilpotent L-variety generated by an F-algebra considered as a vector space. Then an L-variety $\mathcal{M}$ is almost commutative if and only if it is generated by one of the following spaces:

$$A_1 = \langle e_{11}, e_{12} \rangle_F, \qquad A_2 = \langle e_{11}, e_{21} \rangle_F$$

Corollary 3.4.

Let

$$A_3 = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} \middle| a, b \in P \right\},$$

where $\sigma \in \mathrm{Aut} P$, $\sigma \neq 1$ and the field of invariants P^σ is the only maximal subfield of P containing F. L-variety $\mathcal{M}_3 = \mathrm{Var}_L A_3$ contains a noncommutative proper L-subvariety.

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Theorem 3.11 [K., 2021].

Let F be a arbitrary field and $\mathcal{M}$ a L-variety generated by an F-algebra considered as a vector space. Then an L-variety $\mathcal{M}$ is almost Engel if and only if it is generated by one of the following spaces:

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An L-variety $\mathcal{M}$ is called a *minimal nonzero L-variety* (with respect to the inclusion) or an *atom* if for any L-variety $\mathcal{N}$ it follows from inclusion $\mathcal{N} \subseteq \mathcal{M}$ that either $\mathcal{M} = \mathcal{N}$ or $\mathcal{N}$ is the zero L-variety.

In 1956 A. Tarski showed that atoms in the class of rings are generated either by a simple field $GF(p)$ or by a ring with zero multiplication.

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The problem of describing atoms in the class of L-spaces is of interest for study.

Theorem 3.12 [K., 2022].

An L-variety of multiplicative vector spaces over a field $GF(2)$ is an atom iff it coincides with either $\mathcal{M}_0$, or $\mathcal{M}_1$, or $\mathcal{M}_{p(x)}$, where

$$\mathcal{M}_0 = \mathrm{Var}_L\langle xy = 0\rangle,$$
$$\mathcal{M}_1 = \mathrm{Var}_L\langle [x, y] = 0, x + x^2 = 0\rangle,$$
$$\mathcal{M}_{p(x)} = \mathrm{Var}_L\langle [x, y] = 0, x^2y = xy^2, x \cdot p(x) = 0\rangle,$$

$p(x)$ is an irreducible polynomial over the field $GF(2)$.

Note that the identities of the L-variety $\mathcal{M}_{p(x)}$ do not define an atom in the class of linear algebras over the field $GF(q)$.

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Identities of Nonassociative Linear Algebras

Let $\mathfrak{P} = \mathrm{Var}\langle x(yz) = 0 \rangle$ be the variety of left-nilpotent algebras of index 3. This variety of linear algebras was first considered in 1976 by S. V. Polin.

Let V be a vector space and E is the (sub)space of linear transformations of the space V. Consider an algebra $\overline{V} = V \oplus E$, nonzero products of basis elements of of this algebra are given by the rule: $v_i e_{ij} = v_j$ for $v_i \in V$, $e_{ij} \in E$. It is easy to see that $\overline{V} \in \mathfrak{P}$.

As we said earlier, the identities of the algebra $\overline{V} = V \oplus E$ and L-space E are very close related.

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As we said earlier, the identities of the algebra $\overline{V} = V \oplus E$ and L-space E are very close related.

Theorem 4.1 [L'vov, 1978].

The nonassociative polynomial $zf(R_{x_1}, R_{x_2}, \ldots, R_{x_n})$ is an identity of the algebra $\overline{V} = V \oplus E$ iff the associative polynomial $f(x_1, x_2, \ldots, x_n)$ is an identity of the L-space E.

Corollary 4.1.

Let $G = \{f_i(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) | i \in I\} \subseteq F\langle X\rangle$,
$zG = \{zf_i(R_{x_{i_1}}, R_{x_{i_2}}, \ldots, R_{x_{i_k}}) | i \in I\}$.
The set zG is a basis of an identities for the algebra $\overline{V} = V \oplus E$ iff the set G is a basis of an identities for the L-space E.

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Using Corollary 4.1 and the results obtained earlier for multiplicative vector spaces and their identities, we can obtain a number of consequences for nonassociative algebras of the form $\overline{V} = V \oplus E$.

We formulate some corollaries of theorems proved for L-spaces.

We assume that modulo $x(yz) = 0$ the brackets in arbitrary word $x_1x_2 \dots x_k$ are placed according to rule: $x_1x_2 \dots x_k = ((\dots((x_1x_2)x_3)\dots)x_{k-1})x_k$ and the writing $zf(x_1, x_2, \dots, x_n)$ is a short form of the writing of $zf(R_{x_1}, R_{x_2}, \dots, R_{x_n})$.

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Theorem 4.2 [Isaev, K., 2013].

Let F be an infinite field of arbitrary characteristic. The nonassociative algebra $A = \langle v_1, v_2, v_3, v_4\rangle_F \oplus \langle e_{11}, e_{12}, e_{33}, e_{43}\rangle_F$ is an NFB-algebra with a basis of identities:

$$\{x(yz), z\mathrm{St}_3(x,y,t), zx[y,u]v, z[x,y][u,v], z[x,y]z_1z_2\dots z_m[u,v] | m = 1,2,\dots\}.$$

Theorem 4.3 [Isaev, K., 2013].

Let $F = GF(q)$ be an finite field. The nonassociative algebra $A = \langle v_1, v_2, v_3, v_4\rangle_F \oplus \langle e_{11}, e_{12}, e_{33}, e_{43}\rangle_F$ is an NFB-algebra with a basis of identities:

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Let F be an infinite field of characteristic zero. The nonassociative algebra $A = \langle v_1, v_2, e_{11} + e_{12}, e_{22} \rangle_F$ is an NFB-algebra with a basis of identities:

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$$\{x(yz), z(x^{q^2-q+1} - x), z\mathrm{St}_3(x, y, t), z[x, y][u, v], z(x - x^q)(y - y^q), \\ z[x, y](t - t^q), z(x - x^q)[y, t], z[x, y]z_1 z_2 \ldots z_k[u, v] | k = 1, 2, \ldots \}.$$

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The nonassociative algebras constructed in Theorems 4.4 and 4.5 are examples of four-dimensional NFB-algebras. Examples of five-dimensional NFB-algebras were previously known [Maltsev, Parfenov, 1977; L'vov, 1978].

If we put $F = GF(2)$ in Theorem 4.5, then we obtain an example of an NFB-ring containing 16 elements.

Also, having examples of INFB-spaces and SNFB-spaces, we can construct examples of non-associative INFB-algebras and SNFB-algebras, respectively.

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Theorem 4.6 [Isaev, K., 2013].

Let F be an arbitrary field. The algebra $A = V \oplus T_2(F)$ is an SNFB-algebra.

Theorem 4.7 [Isaev, K., 2015].

Let F be an field of characteristic zero, $E_0 = \langle e_{11} + e_{12}, e_{22} \rangle_F$. The algebra $A = V \oplus E_0$ over the field F is an SNFB-algebra.

Corollary 4.2.

Any finite dimensional F-algebra (over the corresponding field F) containing the algebra A as a subalgebra has no finite basis of identities.

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Above, we gave an example of an L-variety that does not have a finite basis of identities and is an union of two Specht varieties.

A similar construction can be constructed for algebras from the variety $\mathfrak{P} = \langle x(yz) = 0 \rangle$.

Let F be a field of characteristic zero, $B_1 = \langle v_1, v_2, e_{11}, e_{12} \rangle_F$, $B_2 = \langle v_1, v_2 + e_{11}, e_{21} \rangle_F$ be F-algebras from the variety $\mathfrak{P}$.

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In 1993, I.P. Shestakov formulated in Dniester notebook a question: do there exist finite dimensional central simple algebras over a field of characteristic zero that do not have a finite basis of identities?

Note that any finite dimensional simple algebra over an algebraically closed field is uniquely determined by their identities up to isomorphism [Shestakov, Zaycev, 2011].

Using the found SNFB-algebras and INFB-algebras, the required example is constructed for an arbitrary field.

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Using the found SNFB-algebras and INFB-algebras, the required example is constructed for an arbitrary field.

Theorem 4.10 [Isaev, K., 2012].

Let $A = \langle 1, v_1, v_2, e_{11}, e_{12}, e_{22}, p \rangle_F$ be an algebra over arbitrary field F, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element are defined by rules: $v_i e_{ij} = v_j$, $v_2 p = 1$. Then the algebra A is a central simple F-algebra and A has no finite basis of identities.

After giving this example, the question was posed about the existence of a finite dimensional simple commutative or anticommutative NFB-algebra.

In the case of characteristic zero, an positive answer to this question was obtained.

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In the case of characteristic zero, an positive answer to this question was obtained.

Theorem 4.11 [K., 2015].

Let $A = \langle 1, v_1, v_2, e_{11}, e_{12}, e_{22}, p \rangle_F$ be an algebra over field F of characteristic zero, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element are defined by rules: $v_i e_{ij} = e_{ij} v_i = v_j$, $v_2 p = p v_2 = 1$. Then the algebra A is a central simple commutative F-algebra and A has no finite basis of identities.

Theorem 4.12 [K., 2017].

Let $A = \langle e, v_1, v_2, e_{11}, e_{12}, e_{22}, p \rangle_F$ be an algebra over field F of characteristic zero, where nonzero products of basis elements are defined by rules: $v_i e_{ij} = -e_{ij} v_i = v_j$, $v_2 p = -p v_2 = e$, $v_i e = -e v_i = v_i$, $e_{ij} e = -e e_{ij} = e_{ij}$, $pe = -ep = p$. Then the algebra A is a simple anticommutative F-algebra and A has no finite basis of identities.

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Let $A = \langle 1, v_1, v_2, e_{11}, e_{12}, e_{22}, p\rangle_F$ be an algebra over field F of characteristic zero, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element are defined by rules: $v_i e_{ij} = e_{ij} v_i = v_j$, $v_2 p = p v_2 = 1$. Then the algebra A is a central simple commutative F-algebra and A has no finite basis of identities.

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Theorems 4.10–4.12 give examples of nonfinitely based simple algebras of dimension seven.

In the case of a field of characteristic zero, the dimension of the algebra from Theorem 4.10 can be decrease to six.

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Theorem 4.13 [K., In proceeding].

Let $A = \langle 1, v_1, v_2, e_{11} + e_{12}, e_{22}, p\rangle_F$ be an algebra over the field F of characteristic zero, where 1 is a unity element of A, and nonzero products of basis elements which is not equal to unity element (taking into account the law of distributive) are defined by rules: $v_i e_{ij} = v_j$, $v_2 p = 1$. Then the algebra A is a central simple F-algebra and A has no finite basis of identities.

Unsolved problems

- Description of NFB-spaces (SNFB-spaces, INFB-spaces).
- Description of minimal nonzero L-varieties of vector spaces.
- Study the structure of the lattice of L-subvarieties.
- Description of almost commutative (almost nilpotent, almost Engel, etc) L-varieties of vector spaces.
- When is $T(G) = L(G)$ for the set G of associative polynomials?
- Construct an example of a not finitely based L-variety, any proper L-subvariety of which is given by a finite number of identities.
- Is there a three dimensional nonassociative NFB-algebra?
- etc.

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Thank you for your attention!