# Musing on exponentiation in groups

Alexei Miasnikov (Stevens Institute)
International Conference CCMAL

September 27th, 2023

# Outline

- Quest for "non-commutative modules"
- Free exponentiation and completions
- Magnus' completions
- Exponentiation in varieties of groups
- Malcev's and Hall exponentiation and completions in nilpotent groups
- Exponentiation in metabelian groups
- Non-standard groups and exponentiation

# Quest for "non-commutative modules"

From abelian groups and vector spaces - to modules over rings.

Nowadays, the theory of modules is an important part of mathematics.

One of the most crucial techniques is the tensor completions.

The main question: How to extend this theory to non-commutative groups?

# Quest for "non-commutative modules"

Every group $G$ has a natural $\mathbb{Z}$-exponentiation ($g \in G, n \in \mathbb{Z}$):

$$g \to g^n.$$

All attempts to define "module" action of a ring $R$ on $G$ are trying to mimic the natural $\mathbb{Z}$-exponentiation on $G$ in such a way that $R$-action is compatible with the algebraic structure of $G$.

# Definitions of R-groups

### Definition

Let $R$ be an associative unitary ring. A group $G$ is called an $R$-group or a group with $R$-exponentiation if it comes equipped with an $R$-action $g \to g^{\alpha}$, where $g \in G, \alpha \in R$, which satisfies the following axioms:

1. $g^1 = g, \;\; g^0 = e, \;\; e^{\alpha} = e.$
2. $g^{\alpha+\beta} = g^{\alpha} g^{\beta}, \;\; g^{\alpha\beta} = (g^{\alpha})^{\beta}.$
3. $(h^{-1}gh)^{\alpha} = h^{-1} g^{\alpha} h.$
4. $(MR$-axiom$) \qquad \forall\, g, h, \alpha \in R \quad gh = hg \longrightarrow (gh)^{\alpha} = g^{\alpha} h^{\alpha}.$

If the action of $R$ on $G$ is faithful, i.e., for any $\alpha \in R$ $G^{\alpha} \neq 1$, then $R$ is called a ring of scalars of $G$.

# Examples of groups with exponentiation

Examples of groups with exponentiation:

- every group $G$ is a $\mathbb{Z}$-group,
- every group of a period $n$ is an $\mathbb{Z}/n\mathbb{Z}$-group;
- $R$-modules over a ring $R$ are abelian $R$-groups,
- divisible groups are $\mathbb{Q}$-groups,
- unipotent groups $G$ over a field $k$ are $k$-groups,
- pro-p-groups are $\mathbb{Z}_p$-groups, where $\mathbb{Z}_P$ is the ring of p-adic integers,
- profinite groups are $\hat{\mathbb{Z}}$-groups, where $\hat{\mathbb{Z}}$ is the completion of $\mathbb{Z}$ in the profinite topology.

# Centroids of groups

A mapping $\phi : G \to G$ is called:

- normal if

$$(h^{-1}gh)^{\phi} = h^{-1}g^{\phi}h, \quad g, h \in G.$$

- a quasi-endomorphism if for any $x, y \in G$ we have:

$$[x, y] = 1 \text{ implies } (xy)^{\phi} = x^{\phi}y^{\phi}.$$

**Theorem [M.]**

The set $\Gamma(G)$ of all normal quasi-endomorphisms of $G$ is an associative ring with 1. It is called the centroid of $G$.

The centroid $\Gamma(G)$ is the generalization of the ring of endomorphisms $End(A)$ of an abelian group $A$ to the non-commutative case.

# Centroids as the maximal rings of scalars

The centroid $\Gamma(G)$ plays an important part in exponentiation in $G$.

**Theorem [M.]**

Let $G$ be a group. Then:

- The centroid $\Gamma(G)$ is a ring of scalars for $G$;
- $\Gamma(G)$ is the largest ring of scalars $G$, i.e., any other ring of scalars of $G$ embeds into $\Gamma(G)$.

Centroids of some groups are known:

- free groups, torsion-free hyperbolic groups, free products, and CSA groups;
- as well as in some nilpotent groups: free nilpotent groups, $UT_n(\mathbb{Z})$.

# Some open problems on centroids in metabelian groups

There is a good approach to compute centroids of finitely generated nilpotent groups via centroids of their associated Lie rings.

It is much less known about centroids of metabelian groups.

Open problems:

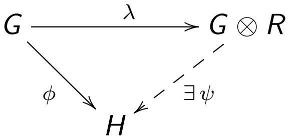
1) Describe the centroid of a free metabelian group.
2) What is the centroid of a metabelian Baumslag-Solitare group $BS(1, n)$?
3) What is the centroid of the wreath product $\mathbb{Z} wr \mathbb{Z}$?

# Tensor completions

## Definition

Let $G$ be a group and $R$ an associative unitary ring. An $R$-group $G \otimes R$ is called a tensor $R$-completion if there is a homomorphism $\lambda : G \to G \otimes R$ such that:

- $\lambda(G)$ $R$-generates $G \otimes R$,
- for any $R$-group $H$ and any homomomorphism $\phi : G \to H$ there exists an $R$-homomorphism $\psi : G \otimes R \to H$ such that $\phi = \psi \circ \lambda$, i.e., the following diagram is commutative




Sometimes the group $G \otimes R$ is denoted by $G^R$.

# Existensce and uniqueness

The following results are standard.

- For every group $G$ and every ring $R$ there exists an $R$-completion $G^R$.
- The completion $G^R$ is unique up to an $R$-isomorphism.
- Let $F$ be a free group. Then $F^R$ is a free $R$-group in the category of $R$ groups.
- The canonical homomorphism $F \to F^R$ is injective, so this $R$-completion is faithful.

# $R$-completions: algebraic structure

Now I will describe algebraic structure of $G^R$ for a very wide class of groups $G$. For this I need a few defintions.

## Definition

A subgroup $H$ of a group $G$ is called conjugately separated or malnormal if $H \cap H^x = 1$ for any $x \in G \smallsetminus H$.

## Definition

A group $G$ is called a CSA-group if all its maximal abelian subgroups are conjugately separated.

Free, torsion-free hyperbolic, and many other groups are CSA. This is a truly large class.

# Tensor extensions of centralizers

From now on for simplicity I will consider only torsion-free groups $G$ and rings $R$ of characteristic zero.

## Tensor extensions of abelian subgroups

Let $G$ be a group and $M$ a maximal abelian subgroup of $G$. Then amalgamated free product

$$G(M, R) = \langle G * (M \otimes R) \mid M = i(M) \rangle,$$

where $i : M \to M \otimes R$ is the canonical embedding, is called the tensor extension of $M$ in $G$ by $R$.

# Algebraic structure of tensor $R$-completions of CSA groups

Theorem [M.-Remeslennikov, 1996]

Let $G$ be a torsion-free CSA group and $R$ a ring of characteristic zero. Then:

- $G^R$ is a union of a chain of tensor extensions of centralizers;
- the canonical homomorphism $G \to G^R$ is inective;
- the group $G^R$ is torsion-free and CSA.

Since $G^R$ is obtained as a chain of amalgamated free products (of a very particular type) Bass-Serre theory gives a powerful approach to various algebraic, geometric, or algorithmic questions about $G^R$.

# Fraisse limits

Theorem [Kharlampovich, M., Sklinos, 2020]

Let $G$ be a torsion-free hyperbolic group then the Lyndon's completion $G^{\mathbb{Z}[t]}$ is a Friasse limit of iterated extension of centralizers of $G$. In particular, $F^{\mathbb{Z}[t]}$ is a Friasse limit of iterated extension of centralizers of $F$.

This result implies various universal and homogeneous properties of $F^{\mathbb{Z}[t]}$.

# Fraisse limits of tensor extensions of centralizers

Theorem [Amaglobeli-M.-Remeslennikov]

Let $G$ be a torsion-free CSA group and $R$ an associative unitary ring of characteristic zero. Then:

- the class $\mathcal{C}$ of iterated tensor $R$-extension of centralizers of $G$ forms a Fraisse category;
- the $R$-completion $G^R$ of $G$ is the Fraisse limit of $\mathcal{C}$.

# How natural are tensor completions?

In the rest of my talk I will discuss:

- How tensor completions appear naturally in groups,
- Their algebraic and algorithmic properties,
- Compare them with other completions.

# Divisible groups

A group $G$ is called divisible (or a $\mathbb{Q}$-group) if every equation of the type

$$x^n = g$$

has a unique solution in $G$ for every $g \in G$ and $n \in \mathbb{Z}$.

By now the theory of divisible groups is more than 100 years old.

For every group $G$ the tensor $\mathbb{Q}$-completion $G^{\mathbb{Q}}$ gives the classical canonical divisible hull of $G$ which is uniquely defined by $G$ (up to $\mathbb{Q}$-isomorphisms).

# Free $\mathbb{Q}$-groups

Let $F = F(X)$ be a free group with basis $X$.

Then $F^{\mathbb{Q}}$ is a free-$\mathbb{Q}$ group with basis $X$, i.e., the free object in the category of divisible groups.

In 1968 G. Baumslag described the structure of $F^{\mathbb{Q}}$ as a union of chain of free root extensions.

A free root extension of an element $g \in G$ is a free product with amalgamation:

$$G *_{g=x^n} \langle x \rangle = \langle G, x \mid g = x^n \rangle.$$

Nowadays, the group $F^{\mathbb{Q}}$ sometimes is called the Baumslag's free $\mathbb{Q}$-group.

# Free $\mathbb{Q}$-groups

The group $F^{\mathbb{Q}}$ enjoys nice algorithmic properties.

Theorem [Kharlampovich-M.]
The Diophantine problem is decidable in $F^{\mathbb{Q}}$.

Decidability of the Diophantine problem means that there is an algorithm that for any finite system of group equations with coefficients in $F^{\mathbb{Q}}$ decides whether or not it has a solution in $F^{\mathbb{Q}}$, and if so, it finds a solution.

# Magnus representation

An integral domain $R$ is a binomial domain if for every $a \in R$ and $n \in \mathbb{N}$ the binomial coefficient $\binom{a}{n}$ is in $R$.

Let $X = \{x_1, \ldots, x_n\}$ and $R\langle\langle x_1, ..., x_n\rangle\rangle$ be the formal power series ring with coefficients in $R$.

If $\Delta_R$ is an ideal generated by $X$ in $R\langle\langle X\rangle\rangle$ then $1 + \Delta_R$ is a group of units in $R\langle\langle X\rangle\rangle$.

One can define an $R$-action on $1 + \Delta_R$ by

$$(1 + f)^{\alpha} = 1 + \Sigma_{n=1}^{\infty} \binom{\alpha}{n} f^n,$$

which turns $1 + \Delta_R$ into an $R$-group.

# Magnus representation

The map $x_i \longrightarrow (1 + x_i)$ can be uniquely extended to an $R$-homomorphism

$$\lambda_R : F^R \longrightarrow 1 + \Delta_R,$$

called the Magnus representation of $F^R$, which is injective on $F$.

Long-standing problem: Is $\lambda_{\mathbb{Q}}$ injective?

Theorem [Jaikin-Zapirain, 2021]

The Magnus representation $\lambda_R : F^R \longrightarrow 1 + \Delta_R$ is injective for any binomial domain $R$.

Corollary. The group $F^R$ is residually torsion-free nilpotent.

# Magnus representation

The map $x_i \longrightarrow (1 + x_i)$ can be uniquely extended to an $R$-homomorphism

$$\lambda_R : F^R \longrightarrow 1 + \Delta_R,$$

called the Magnus representation of $F^R$, which is injective on $F$.

Long-standing problem: Is $\lambda_{\mathbb{Q}}$ injective?

Theorem [Jaikin-Zapirain, 2021]

The Magnus representation $\lambda_R : F^R \longrightarrow 1 + \Delta_R$ is injective for any binomial domain $R$.

Corollary. The group $F^R$ is residually torsion-free nilpotent.

# Magnus representation for CSA groups

One can construct a similar representation

$$\mu_{G,R} : G^R \to 1 + \bar{\Delta}_R,$$

for a torsion-free CSA group $G$. Here the algebra $R\langle\langle X \rangle\rangle$ is replaced with the $\Delta$-adic completion $R[G]^*$ of the group algebra $R[G]$ with respect to the augmentation ideal $\Delta$ in $R[G]$, and $\bar{\Delta}_R$ is an augmentation ideal in $R[G]^*$.

## Problem

Compute the kernel of $\mu_{G,R}$ for binomial domains $R$ and torsion-free CSA groups $G$. In particular, is $\mu_{G,R}$ injective for a limit group $G$?

# $\mathbb{Q}$-completions of limit groups

Theorem [Jaikin-Zapirain, 2021]

The Magnus representation $\mu_{G,\mathbb{Q}} : G^{\mathbb{Q}} \to 1 + \bar{\Delta}_{\mathbb{Q}}$ is injective for any limit group $G$.

Corollary. The $\mathbb{Q}$-completion $G^{\mathbb{Q}}$ of a limit group $G$ is residually torsion-free nilpotent.

Open problem

Is the Diophantine problem decidable in $G^{\mathbb{Q}}$ for a limit group $G$?

# Pro-p-completions

For a group $G$ denote by $\widehat{G}_p$ the pro-p-completion of $G$, i.e., the inverse limit of all finite p-quotients of $G$.

As we mentioned above $\widehat{G}_p$ is a $\mathbb{Z}_p$-group.

If $F(X)$ is a free group with basis $X$ then $\widehat{F(X)}_p$ is a free pro-p group with basis $X$.

Theorem [Jaikin-Zapirain, 2021]

Let $X$ be a finite set then the canonical $\mathbb{Z}_p$-homomorphism

$$F(X)^{\mathbb{Z}_p} \to \widehat{F(X)}_p$$

is injective. In particular, $F(X)^{\mathbb{Z}_p}$ is residually torsion-free nilpotent.

# Open problems for tensor $\mathbb{Z}_p$-completions

## Open problem

Is the canonical $\mathbb{Z}_p$-homomorphism $G^{\mathbb{Z}_p} \to \widehat{G}_p$ injective for a limit group $G$?

## Open problem

Is the Diophantine problem decidable in $G^{\mathbb{Z}_p}$ (with coefficients from $G$) for a limit group $G$?

## Old open problem

Is the Diophantine problem decidable in $\widehat{G}_p$ (with coefficients from $G$) for a limit group $G$? In particular, is it decidable in the pro-p-completion $\widehat{F}_p$ for a free group $F$?

# Open problems for tensor $\mathbb{Z}_p$-completions

**Open problem**

Is the canonical $\mathbb{Z}_p$-homomorphism $G^{\mathbb{Z}_p} \to \widehat{G}_p$ injective for a limit group $G$?

**Open problem**

Is the Diophantine problem decidable in $G^{\mathbb{Z}_p}$ (with coefficients from $G$) for a limit group $G$?

**Old open problem**

Is the Diophantine problem decidable in $\widehat{G}_p$ (with coefficients from $G$) for a limit group $G$? In particular, is it decidable in the pro-p-completion $\widehat{F}_p$ for a free group $F$?

# Open problems for tensor $\mathbb{Z}_p$-completions

## Open problem

Is the canonical $\mathbb{Z}_p$-homomorphism $G^{\mathbb{Z}_p} \to \widehat{G}_p$ injective for a limit group $G$?

## Open problem

Is the Diophantine problem decidable in $G^{\mathbb{Z}_p}$ (with coefficients from $G$) for a limit group $G$?

## Old open problem

Is the Diophantine problem decidable in $\widehat{G}_p$ (with coefficients from $G$) for a limit group $G$? In particular, is it decidable in the pro-p-completion $\widehat{F}_p$ for a free group $F$?

# Remeslennikov's problem

The following problem attracted a lot of attention during the last couple of decades. It it is open still.

Qpen Problem [Remeslennikov]

Let $F$ be a free non-abelian group of finite rank and $G$ a finitely generated residually finite group. Is it true that

$$\hat{F} \simeq \hat{G} \Longleftrightarrow F \simeq G?$$

In view of the previous results it would be interesting to look at seemingly easier but related problem problems.

# Rigidity problems for exponentiation

First rigidity problem: Let $R$ and $S$ be two commutative unitary rings of characteristic 0 and $F$ a free group. Is it true that

$$F^R \simeq F^S \Longrightarrow R \simeq S?$$

One can start with the case when $R$ and $S$ are fields.

Second rigidity problem: Let $F$ be a free group and $G$ a torsion-freeCSA group (say torsion-free hyperbolic) and $R$ an integral domain. Is it true that

$$F^R \simeq G^R \Longrightarrow F \simeq G?$$

# Rigidity problems for exponentiation

First rigidity problem: Let $R$ and $S$ be two commutative unitary rings of characteristic 0 and $F$ a free group. Is it true that

$$F^R \simeq F^S \Longrightarrow R \simeq S?$$

One can start with the case when $R$ and $S$ are fields.

Second rigidity problem: Let $F$ be a free group and $G$ a torsion-freeCSA group (say torsion-free hyperbolic) and $R$ an integral domain. Is it true that

$$F^R \simeq G^R \Longrightarrow F \simeq G?$$

# Lyndon'sgroup $F^{\mathbb{Z}[t]}$

In 1960 Lyndon introduced a free $\mathbb{Z}[t]$-group $F^{\mathbb{Z}[t]}$ to study equations and algebraic geometry in free groups $F$.

He treated elements of $F^{\mathbb{Z}[t]}$ as parametric words, where together with the standard multiplication one is allowed to take exponentiation by polynomials from $\mathbb{Z}[t]$, such as

$$(x^{f(t)}y^{g(t)})^{h(t)}u^{s(t)},$$

modulo the congruence generated by the consequences of the axioms.

# Algebraic sets in $F$

His idea was to describe the solution sets $A$ of finite system of equations $S(x_1, \ldots, x_n, F) = 1$ in $F$ as finite unions of parametric sets

$$A = P_1 \cup \ldots \cup P_n,$$

where $P_i$ is the set of all values of some parametric word $w_i \in F^{\mathbb{Z}[t]}$ under all specialization homomorphisms $\phi_n : F^{\mathbb{Z}[t]} \to F$ induced by the homomorphism $\mathbb{Z}[t] \to \mathbb{Z}$ that maps $t \to n$, $n \in \mathbb{Z}$.

In particular, the parametric set defined by a parametric word $u^t$, where $u \in F$ is the cyclic subgroup generated by $u$.

It follows that equation $xu = ux$ in $F$, where $u \in F$ is not a proper power is described precisely by trhe parametric word $u^t$.

# Algebraic sets in $F$

Lyndon showed that the solution sets of one-variable equations are indeed finite unions of parametric sets.

In general his idea does not hold, but his intuition was not far off as the following result shows.

## Theorem [Kharlampovich-M.]

Every algebraic set (the solution set of a finite system of equations) over $F$ can be obtained as the Zariski closure of a finite union of parametric sets.

We obtained a much more precise description of algebraic sets, but there is no time to explain it here.

# Discriminating homomorphisms

Lyndon's proofs were technically quite challenging because he did not have a clear algebraic description of the structure of the group $F^{\mathbb{Z}[t]}$.

Nevertheless, he proved that the set of specialization homomorphisms

$$\Phi = \{\phi_n : F^{\mathbb{Z}[t]} \to F \mid n \in \mathbb{Z}\}$$

discriminates $F^{\mathbb{Z}[t]}$ into $F$, i.e., for any finite set of elements $E \subset F^{\mathbb{Z}[t]}$ there is $\phi_n \in \Phi$ such that the restriction of $\phi_n$ on $E$ is injective.

# Elements of algebraic geometry and coordinate groups

Like in the classical algebraic geometry to understand solution sets of finite systems of equations in a group one has to study the coordinate groups of such systems.

Zariski topology (generated by the algebraic sets as prebasis of closed sets) over free groups is Noetherian, hence every algebraic set is a finite union of irreducible ones.

This implies that it suffices to study only the coordinate groups of the irreducible algebraic sets.

# Uniformization theorems

The following result is rather general, but I state it just for free groups.

## Theorem [Baumslag-M.-Remeslennikov]

Let $G$ be a finitely generated group. Then the following conditins are equivalent:

- $G$ is the coordinate group of an irreducible algebraic set over $F$;
- $G$ is universally equivalent to $F$;
- $G$ is discriminated by $F$.

Combining this with Lyndon's result on discrimination of $F^{\mathbb{Z}[t]}$ one immediately gets a large source of the coordinate groups of irreducible algebraic sets over $F$ - all finitely generated subgroups of $F^{\mathbb{Z}[t]}$.

# Subgroups of $F^{\mathbb{Z}[t]}$

By that time Bass-Serre theory was already developed. By design, it allows one to describe subgroups of free constructions as fundamental groups of graphs of groups.

In particular, it gives description of finitely generated subgroups of $F^{\mathbb{Z}[t]}$ as the fundamental groups of particularly nice graphs of groups.

How about other groups discriminated by $F$, which are not subgroups of $F^{\mathbb{Z}[t]}$?

# Subgroups of $F^{\mathbb{Z}[t]}$

By that time Bass-Serre theory was already developed. By design, it allows one to describe subgroups of free constructions as fundamental groups of graphs of groups.

In particular, it gives description of finitely generated subgroups of $F^{\mathbb{Z}[t]}$ as the fundamental groups of particularly nice graphs of groups.

How about other groups discriminated by $F$, which are not subgroups of $F^{\mathbb{Z}[t]}$?

## Coordinate groups: the last step

Theorem [Kharlampovich and M.]

Every finitely generated group discriminated by $F$ embeds into $F^{\mathbb{Z}[t]}$.

This is technically a very demanding result that requires some essential development of Makanin-Razborov technique on solving equations in free groups, in particular, a novel description of solution sets of systems of equations in free groups via triangular quasi-quadratic systems.

# The Diophantine problem

The group $F^{\mathbb{Z}[t]}$ is very well studied. I just mention two results.
The main approach to problems on finitely generated subgroups of $F^{\mathbb{Z}[t]}$ is via Bass-Serre theory which gives a description of finitely generated subgroups as the fundamental groups of particularly nice graphs of groups.

Theorem [Kharlampovich-M.]

The Diophantine problem is decidable in $F^{\mathbb{Z}[t]}$.

# Tarski problems in free groups

Tarski problems for free groups $F$ were solved by Kharlampovich and Miasnikov, and, independently, by Sela:

- $Th(F)$ is decidable
- All nonabelian free groups have the same first-order theory ($Th(F_n) = Th(F_m)$ for all $m, n \geq 2$)
- $Th(F)$ has effective quantifier elimination to $\forall\exists$-formulas.

# First-order classification problem

First-order classification problem for $F$

Describe all groups which are first-order equivalent to a free group $F$.

A group $G$ is called elementarily free if $G \equiv F$, where $F$ is a free non-abelian group.

The main focus is on countable elementarily free groups, it is old and difficult problem.

# First-order classification for free groups

**Theorem [Kharlampovich-M., Sela, 2006]**

Let $F$ be a nonabelain free group. Regular NTQ groups ($\omega$-residually free towers, hyperbolic towers) are exactly the finitely generated elementarily free groups.

# Countable elementary free groups

Theorem [Kharlampovich-Natoli]

Let $G$ be a countable elementary free group in which all abelian subgroups are cyclic. Then $G$ is a union of a chain of finitely generated elementary free groups.

However, $\mathbb{Z} * (\mathbb{Z} + \mathbb{Q}) \equiv \mathbb{Z} * \mathbb{Z}$ which shows that not every countable elementary free group is a union of a chain of regular NTQ groups.

Recently a new method was developed to construct "non-standard version" of groups.

They provide a completely new type of elementarily free groups.

# Non-standard models of $F$

Recall that a non-standard model of arithmetic is any ring $\tilde{\mathbb{Z}}$ such that $\mathbb{Z} \equiv \tilde{\mathbb{Z}}$.

**Theorem**

For every non-standard countable model of arithmetic $\tilde{\mathbb{Z}}$ there exists a unique non-standard free group $F(\tilde{\mathbb{Z}})$ such that:

- $F$ is the "standard part" of $F(\tilde{\mathbb{Z}})$;
- $F$ is an elementary subgroup of $F(\tilde{\mathbb{Z}})$;
- not only $F \equiv F(\tilde{\mathbb{Z}})$, but $F$ and $F(\tilde{\mathbb{Z}})$ are equivalent in a much stronger logic, a variation of the weak second order logic. They are strong models of the first-order theory of $F$.

# Non-standard models of $F$

Properties of non-standard fee groups:

- The groups $F(\tilde{\mathbb{Z}})$ are $\tilde{\mathbb{Z}}$-groups.
- If $X$ is a basis of $F$ then $X$ is a "non-standard" basis of $F(\tilde{\mathbb{Z}})$.
- The identical map $X \to X$ extends to a $\tilde{\mathbb{Z}}$-homomorphism

$$\nu_{\tilde{\mathbb{Z}}} : F^{\tilde{\mathbb{Z}}} \to F(\tilde{\mathbb{Z}}).$$

Conjecture:

The map $\nu_{\tilde{\mathbb{Z}}} : F^{\tilde{\mathbb{Z}}} \to F(\tilde{\mathbb{Z}})$ is injective.

If this is true then we know quite a bit about the algebraic structure of the non-standard free groups $F(\tilde{\mathbb{Z}})$.

# Free $R$-groups and their $R$-subgroups

The Main Theme: study $R$-subgroups of a free $R$-group $F^R$.

Open-ended problem 1: Develop a theory of free actions on "free" $R$-trees.

Note that for an ordered abelian group $\Lambda$ one can define $\Lambda$-trees and $\Lambda$-hyperbolic spaces.

However, these may not be "universal" objects in the corresponding categories.

# Free $R$-actions on free $R$-trees

One possible way to look for universal objects:

- Take an abelian ordered group $\Lambda$ and form its tensor algebra $T(\Lambda)$,
- Take a $\Lambda$-tree $\Gamma$ and consider its tensor $T(\Lambda)$-completion $\Gamma \otimes_\Lambda T(\Lambda)$.

In particular,

- Let $\mathbb{Z}$ be an infinite cyclic group.
- Then $T(\mathbb{Z}) \simeq \mathbb{Z}[t]$ is the ring of integer polynomials in one variable.
- Take a $\mathbb{Z}$-tree (simplicial tree) $\Gamma$ and form $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[t]$.

# Free $\mathbb{Z}$-actions on free $\mathbb{Z}$-trees

Question:

Are limits groups precisely the finitely generated groups that act freely on $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ fro some $\mathbb{Z}$-tree (simplicial tree) $\Gamma$?

# $R$-free constructions of $R$-groups and their $R$-subgroups

In the category of $R$-groups one can define free $R$-products with amalgamation and HNN $R$-extensions.

Open-ended problem 2: Study these free $R$-constructions.

Open-ended problem 3: Develop an analog of Bass-Serre theory for $R$-groups.

# Exponentiation in varieties of groups

For $g, h \in G$ and $\alpha \in R$, the element

$$(g, h)_\alpha = h^{-\alpha} g^{-\alpha} (gh)^\alpha$$

is called the $\alpha$-commutator of the elements $g$ and $h$. Clearly,

$$(gh)^\alpha = g^\alpha h^\alpha (g, h)_\alpha.$$

A normal $R$-subgroup $H$ of an $R$-group $G$ is an $R$-ideal if for any $g, h \in G$ fand $\alpha \in R$

$$[g, h] \in H \Longrightarrow (g, h)_\alpha \in H.$$

- Kernels of $R$-homomorphisms are $R$-ideals;
- If $H$ is an $R$-ideal of $G$, then $G/H$ is an $R$-group.

# Varieties of $R$-groups

Let $W$ be a set of $R$-words in the group language extended by exponentiation operations $f_\alpha, \alpha \in R$.

A word $w(x_1, \ldots, x_n) \in W$ is an $R$-identity in an $R$-group $G$ if $w(g_1, \ldots, g_n) = 1$ for all $g_1, \ldots, g_n \in G$.

The set $W$ determines a variety of $R$-groups $\mathcal{V}_R(W)$ consisting of all $R$-groups satisfying the identities from $W$.

# Varieties of nilpotent and solvable $R$-groups

The left-norm commutator $[x_1, \ldots, x_{c+1}]$ of weight $c$ defines the variety of all $R$-groups which are nilpotent of class $\leq c$.

Here

$$[x_1, \ldots, x_{c+1}] = [[x_1, \ldots, x_c], x_{c+1}].$$

Put $\delta_0 = x$ and

$$\delta_{n+1}(x_1, \ldots, x_{2^{n+1}}) = [\delta_n(x_1, \ldots, x_{2^n}), \delta_n(x_{2^n+1}, \ldots, x_{2^{n+1}})].$$

Then $\delta_c(\bar{x})$ defines the variety of all $R$-groups which are solvable of class $\leq c$.

# Free $R$-groups in the variety

From universal algebra we know that for any set $X$ the variety $\mathcal{V}_R(W)$ has free $R$-groups with basis $X$ denoted $F_{W,R}(X)$.

**Definition**
The $R$-ideal $W(G)$ in $G$ generated by the values of all words in a set $W$ is called the *$W$-verbal ideal in $G$*.

**Free $R$-groups in varieties:**
A free group in the variety $\mathcal{V}(W)$ of $R$-groups defined by a set of $R$-words $W$ is $R$-isomorphic to

$$F_{W,R}(X) = F^R(X)/W(F^R(X)),$$

where $F^R(X)$ is an absolutely free $R$-group with basis $X$.

# Tensor completions in varieties

Let $S$ be a subring of $R$, $W$ a set of $R$-words.

For a group $G \in \mathcal{V}_S(W)$ one can define its tensor $R$-completion $G_W^R \in \mathcal{V}_R(W)$ by the corresponding universal property.

From universal algebra $G_W^R$ exists and it is unique.

Furthermore, for $G \in \mathcal{V}_S(W)$

$$G_W^R \simeq G^R / W(G^R).$$

Main question: What is the algebraic structure of $G_W^R$?

# Tensor completions in the classical varieties

Let $W = W(X)$ be a set of the usual group words (i.e., $\mathbb{Z}$-words) and $R$ a unitary ring of characteristic zero, so $\mathbb{Z} \leq R$.

Then $\mathcal{V}_{\mathbb{Z}}(W)$ is the classical variety $\mathcal{V}(W)$ of groups defined by $W$.

For $G \in \mathcal{V}(W)$ the group $G_W^R$ is the tensor $R$ completion of $G$ relative to $W$.

Main questions:

- how natural is this tensor completion $G_W^R$ for varieties $W$ of nilpotent and solvable groups?
- what is the algebraic structure of $G_W^R$?

# Faithfulness of tensor completions of nilpotent groups

Let $G$ be a torsion-freenilpotent group and $R$ an integral domain, so $\mathbb{Z} \leq R$.

Then the following hold:

- The canonical homomorphism $G \to G^R$ is injective.
- Tensor $R$-completions of residually torsion-free nilpotent group are faithful.
- Tensor $R$-completions of free solvable groups are faithful.

# Mal'cev and Hall's exponentiation in nilpotent groups

### Definition

Let $G$ be a torsion-freenilpotent group and $R$ a binomial domain. A group $G$ is called a Hall's $R$-group or a group with Hall's $R$-exponentiation if it comes equipped with an $R$-action $g \to g^{\alpha}$, where $g \in G, \alpha \in R$, which satisfies the following axioms:

1. $g^1 = g, \;\; g^0 = e, \;\; e^{\alpha} = e.$
2. $g^{\alpha+\beta} = g^{\alpha} g^{\beta}, \;\; g^{\alpha\beta} = (g^{\alpha})^{\beta}.$
3. $(h^{-1}gh)^{\alpha} = h^{-1} g^{\alpha} h.$
4. $g_1^{\alpha} \dots g_n^{\alpha} = (g_1 \dots g_n)^{\alpha} t_2^{\binom{\alpha}{2}} \dots t_c^{\binom{\alpha}{c}},$
   where $c$ is the nilpotency class of $G$, and $t_k = t_k(g_1, \dots, g_n)$ are Petresco words.

# Mal'cev and Hall's completions of nilpotent groups

Every torsion-free finiteley generated nilpotent group $G$ has a Mal'cev basis, i.e., a tuple of elements $u_1, \ldots, u_n \in G$ such that every $g \in G$ has a unique representation in the form

$$g = u_1^{t_1} \ldots u_n^{t_n}, \quad t_i \in \mathbb{Z}$$

Exponents $t_i$ are called Malcev's coordinates of $g$ denoted $t(g) = (t_1, \ldots, t_n)$.

In terms of coordinates $t(g)$ multiplication in $G$ is given by some polynomials $f_i(\bar{x}, \bar{y})$, $i = 1, \ldots, n$ with rational coefficients in such a way that for $g, h \in G$

$$t_i(gh) = f_i(t(g), t(h)).$$

The standard $\mathbb{Z}$-exponentiation in $G$ is also defined by some polynomials.

# Mal'cev and Hall's completions of nilpotent groups

The Hall's completion $G \otimes_H R$ of $G$ by a binomial domain $R$:

- as a set consists of all tuples $R^n$ ($n$ is the length of a Mal'cev's base),
- multiplication on $R^n$ is defined via the polynomials $f_i(\bar{x}, \bar{y})$.
- $R$-exponentiation on $R^n$ is defined by the same polynomials as in $G$.

$G \otimes_H R$ is a nilpotent $R$-group and $G$ naturally embeds into it.

# MR and Hall's completions of nilpotent groups

Let $\mathcal{N}_c$ be a variety of nilpotent groups of class $c$ defined by the left-normed commutator $[x_1, \ldots, x_{c+1}]$.

Now for a torsion-free nilpotent group $G \in \mathcal{N}_c$ and a binomial domain $R$ we have two $R$-completions:

- $R$-completion of $G$ in the variety $\mathcal{N}_c$, which I now denote by $G \otimes R$.
- Hall's $R$-completion $G \otimes_H R$.

Both are $R$-groups, but the Hall's axiom 4) is more restrictive.

How do they relate to each other?

# MR and Hall's completions of nilpotent groups

The group $G$ embeds in both $G \otimes R$ and $G \otimes_H R$ and $R$-generates both of them.

Therefore the identical map $G \to G$ extends to an $R$-epimorphism

$$\lambda_{G,R} : G \otimes R \to G \otimes_H R.$$

**Problem**

Describe the kernel of $\lambda_{G,R} : G \otimes R \to G \otimes_H R$.

# MR and Hall's completions of $UT_3(\mathbb{Z})$

Let $H = UT_3(\mathbb{Z})$ be the Heizenberg group, which is also a free 2-nilpotent group of rank 2.

Theorem [Amaglobeli - Remeslennikov]

Let $R = \mathbb{Q}[t]$ or $R = \mathbb{Q}(t)$. Then

$$H \otimes R \simeq H \otimes_H R \times D,$$

where $D$ is a free $R$-module of countable rank and the direct product is of abstract groups (not $R$-groups).

Note, that to prove this result the authors constructed precisely the group $H \otimes R$.

# MR and Hall's completions of nilpotent groups of class 2

Theorem [ Amaglobeli - M. - Remeslennikov]

Let $N$ be a free nilpotent of class 2 group and $K$ a field of characteristic 0. Then

$$N \otimes K \simeq N \otimes_H K \times D,$$

where $D$ is a $K$-vector space of dimension $|N \otimes_H K|$ and the direct product is of abstract groups.

Corollary

- $N \otimes_H K$ naturally embeds into $N \otimes K$.
- $N \otimes_H K$ is a retract of $N \otimes K$.
- $Ker(\lambda_{N,K}) = D$ is a central subgroup of $N \otimes K$.

# MR and Hall's completions of nilpotent groups

Conjecture: Prove that the theorem above holds for an arbitrary torsion-free nilpotent of class 2 group $G$, i.e., for a field $K$

$$G \otimes K \simeq N \otimes_H K \times D,$$

where $D$ is a $K$-vector space of a suitable dimension.

Very open question: is the same true for an arbitrary torsion-free nilpotent group $G$?

# Algebraic structure of solvable and nilpotent $R$-groups

In group theory one can define solvability either through identities or via series of commutants.

Likewise, nilpotency can be defined either through identities or via central series.

It is not clear whether these definitions are equivalent in the class of $R$-groups.

To address the problem one needs to study $\alpha$-commutators identities.

# Commutators and $\alpha$-commutators

The standard commutator $[x, y] = x^{-1}y^{-1}xy$ and $xy = yx[x, y]$.

For $\alpha \in R$ the $\alpha$-commutator $(x, y)_\alpha$ is defined by

$$(x, y)_\alpha = y^{-\alpha}x^{-\alpha}(xy)^\alpha$$

so $(xy)^\alpha = x^\alpha y^\alpha (x, y)_\alpha$.

Note that $[x, y] = (y^{-1}, x^{-1})_{-1}$.

In group theory the whole commutator calculus was developed to deal with commutator identities.

There is, it seems, some $\alpha$-commutator calculus for $R$-groups. For example,

$$[g^\alpha, f] = [g, f]^\alpha \big(g, [g, f]\big)_\alpha.$$

# $R$-commutant

Let $G$ be an $R$-group. Then the $R$-subgroup

$$(G,G)_R = \langle (g,h)_\alpha \mid g,h \in G, \alpha \in R \rangle_R$$

$R$-generated in $G$ by all $\alpha$-commutators is called the $R$-commutant of $G$.

## Properties of $(G,G)_R$

- $(G,G)_R$ is the verbal subgroup of $G$ defined by the word $x^{-1}y^{-1}xy$.
- $(G,G)_R$ is the smallest $R$-ideal $H$ in $G$ such that $G/H$ is abelian.

# $R$-commutant series

For an $R$-group $G$ put $G = G^{(0,R)}$ and

$$G^{(n+1,R)} = (G^{(n,R)}, G^{(n,R)})_R.$$

This gives the $R$-commutant series:

$$G = G^{(0,R)} \geq G^{(1,R)} \geq G^{(2,R)} \geq \cdots \geq G^{(n,R)} \geq \cdots$$

**Definition**

An $R$-group $G$ is called $R$-solvable if $G^{(n,R)} = 1$ for some $n$.

# Two definitions of solvable $R$-groups

Now we have two different types of higher commutants:

- The $n$th $R$-commutant $G^{(n,R)}$,
- The $R$-ideal $id_R(G^{(n)})$, where $G^{(n)}$ is the verbal subgroup in $G$ generated by the word $\delta_n(\bar{x})$.

Clearly, $G^{(n,R)} \geq id_R(G^{(n)})$.

Open problem: Is it true that $G^{(n,R)} = id_R(G^{(n)})$ for any $n$?

Note, that $(G, G)_R = id_R([G, G])$, hence it is true for $n = 1$.

If $G^{(n,R)} = id_R(G^{(n)})$ for any $n$ then $R$-solvable groups are precisely the solvable $R$-groups.

## Lower central series of verbal ideals

Let $G$ be an $R$-group.

Put $\gamma_0(G) = G$ and define

$$\gamma_{n+1}(G) = id_R([G, \gamma_n(G)]),$$

i.e., $\gamma_{n+1}(G)$ is the $R$-ideal generated by $[G, \gamma_n(G)]$.

This gives a series of $R$-ideals

$$\gamma_0(G) = G \geq \gamma_1(G) \geq \dots \gamma_n(G) \geq \dots$$

Definition An $R$-group $G$ is called $R$-nilpotent if there is $n$ such that $\gamma_{n+1}(G) = 1$. The smallest such $n$ is called the nilpotency class of $G$.

Let $\mathcal{N}_{c,R}$ be the class of all $R$-nilpotent groups of class $c$.

## $R$-lower central series

Let $G$ be an $R$-group.

Put $\bar{\gamma}_0(G) = G$ and define

$$\bar{\gamma}_{n+1}(G) = (G, \bar{\gamma}_n(G))_R.$$

as the $R$-commutant of $G$ and $\bar{\gamma}_n(G)$.

This gives a series of $R$-ideals

$$\bar{\gamma}_0(G) = G \geq \bar{\gamma}_1(G) \geq \dots \bar{\gamma}_n(G) \geq \dots$$

which we call lower $R$-central series of $G$.

Definition An $R$-group $G$ is called lower $R$-nilpotent if there is $n$ such that $\bar{\gamma}_{n+1}(G) = 1$. The smallest such $n$ is called the nilpotency class of $G$.

# Lower central series of verbal ideals

Let $G$ be an $R$-group.

Put $\gamma_0(G) = G$ and define

$$\gamma_{n+1}(G) = id_R([G, \gamma_n(G)]),$$

i.e., $\gamma_{n+1}(G)$ is the $R$-ideal generated by $[G, \gamma_n(G)]$.

This gives a series of $R$-ideals

$$\gamma_0(G) = G \geq \gamma_1(G) \geq \ldots \gamma_n(G) \geq \ldots$$

Definition An $R$-group $G$ is called $R$-nilpotent if there is $n$ such that $\gamma_{n+1}(G) = 1$. The smallest such $n$ is called the nilpotency class of $G$.

# Two definitions of nilpotent $R$-groups

Now we have two different types of central series in $G$:

- $\gamma_0(G) = G \geq \gamma_1(G) \geq \ldots \gamma_n(G) \geq \ldots,$
- $\bar{\gamma}_0(G) = G \geq \bar{\gamma}_1(G) \geq \ldots \bar{\gamma}_n(G) \geq \ldots$

Clearly, $\bar{\gamma}_n(G) \geq \gamma_n(G)$ for each $n$.

Open problem: Is it true that $\bar{\gamma}_n(G) = \gamma_n(G)$ for each $n$?

If $\bar{\gamma}_n(G) = \gamma_n(G)$ for any $n$ then $R$-nilpotent groups are precisely the nilpotent $R$-groups.

# Two definitions of nilpotent $R$-groups

Theorem [Amaglobeli-Nadiradze-M.]

For any $R$-group $G$

$$\bar{\gamma}_2(G) = \gamma_2(G).$$

Hence the two definitions of the 2-nilpotent $R$-groups are equivalent.

# Exponentiation in metabelian groups

For nilpotent groups $N$ there are two types of $R$-completions of $N$:

- free tensor $R$-completion $N \otimes_{MR} R$,
- Hall $R$-completion $N \otimes_H R$.

They are different but close which allows one to study $N \otimes_{MR} R$ via $N \otimes_H R$.

Is the situation similar in the class of solvable, say metabelian groups?

# Free metabelian groups

Let $G$ be a free metabelian group with basis $X = \{x_1, \ldots, x_n\}$.

The group $G$ acts by conjugation on $G'$, which gives an action of the abelianization $\bar{G} = G/G'$ on $G'$.

This action extends by linearity to an action of the group ring $\mathbb{Z}\bar{G}$ on $G'$ and turns $G'$ into a $\mathbb{Z}\bar{G}$-module.

$\mathbb{Z}\bar{G}$ can be viewed as the Laurent polynomial ring

$$A = \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}],$$

where $a_i = x_i G'$.

# Normal Forms for $G$

Lemma [M.- Romankov]

Every element $u \in G'$ can be uniquely presented as a product

$$u = \Pi_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)},$$

where $\beta_{ij}(a_1, \ldots, a_i) \in \mathbb{Z}[a_1, a_1^{-1}, \ldots, a_i, a_i^{-1}] \leq \mathbb{Z}\bar{G}$.
In particular, every element $g \in G$ can be uniquely presented as a product

$$g = x_1^{\gamma_1} \ldots x_n^{\gamma_n} \Pi_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)},$$

where $\gamma_i \in \mathbb{Z}$ and $\beta_{ij}$ as above.

# Groups elementarily equivalent to $G$

**Theorem [Kharlampovich-M.]**

Let $G$ be a free metabelian group with basis $X$. Then for an arbitrary group $H$ $H \equiv G$ if and only if $H$ is the non-standard version $G(\tilde{\mathbb{Z}})$ of $G$ for some non-standard arithmetic $\tilde{\mathbb{Z}}$.

This is based on

**Theorem [Kharlampovich-M. - Sohrabi]**

Let $G$ be a free metabelian group of finite rank $r \geq 2$. Then $G$ is regularly bi-interpretable with $\mathbb{Z}$.

One can say a lot on the algebraic structure of $G(\tilde{\mathbb{Z}})$. For us now the following is of interest.

Corollary

- $G$ is an elementary subgroup of $G(\tilde{\mathbb{Z}})$.
- $G(\tilde{\mathbb{Z}})$ is non-standardly generated by $X$.
- The group $G(\tilde{\mathbb{Z}})$ is a metabelian $\tilde{\mathbb{Z}}$-group.

Question: How far $G(\tilde{\mathbb{Z}})$ is from $G^{\tilde{\mathbb{Z}}}$?

# Non-standard free metabelian groups

The group $G^{\tilde{\mathbb{Z}}}$ is a free metabelian $\tilde{\mathbb{Z}}$-group with basis $X$.

Hence the identical map $X \to X$ extends to a $\tilde{\mathbb{Z}}$-homomorphism

$$\lambda_{\tilde{\mathbb{Z}}} : G^{\tilde{\mathbb{Z}}} \to G(\tilde{\mathbb{Z}}).$$

Theorem [Kharlampovich-M.]

The homomorphism $\lambda_{\tilde{\mathbb{Z}}} : G^{\tilde{\mathbb{Z}}} \to G(\tilde{\mathbb{Z}})$ is injective.

Now we know much more on the structure of the $\tilde{\mathbb{Z}}$-subgroup $\langle G \rangle_{\tilde{\mathbb{Z}}}$ in $G(\tilde{\mathbb{Z}})$ generated by $G$.

# More on the structure of $G(\tilde{\mathbb{Z}})$

Let $\tilde{\mathbb{Z}}\bar{G}_{NS}$ be a non-standard ring of Laurent polynomials. (Miasnikov, Nikolaev)

# Structure of $G(\tilde{\mathbb{Z}})$

Theorem [Kharlampovich-M.]

Let $H = G(\tilde{\mathbb{Z}})$ be a non-standard free metabelian group. Then:

- $H/H'$ is a free $\tilde{\mathbb{Z}}$-module.
- $H'$ is a module over $\tilde{\mathbb{Z}}\bar{G}_{NS}$ with generators $\{[x_i, x_j] \mid 1 \leq j < i \leq n\}$.
- Elements $h \in H$ can be uniquely represented as

$$h = x_1^{\tilde{\gamma}_1} \ldots x_n^{\tilde{\gamma}_n} \Pi_{1 \leq j < i \leq n} [x_i, x_j]^{\beta_{ij}(a_1, \ldots, a_i)},$$

  where $\tilde{\gamma}_i \in \tilde{\mathbb{Z}}$, $\beta_{ij}(a_1, \ldots, a_i) \in \tilde{\mathbb{Z}}\bar{G}_{NS}$.
- Multiplication of the normal forms is explicitly given by some functions (more on this below).

# $A$-metabelian groups

Let $A$ be a discretely ordered commutative ring (no element between 0 and 1) and $K$ a multiplicative $A$-module with generators $a_1, \ldots, a_n$.

Let $A(K)$ be the group $A$-algebra over $K$.

Extend $A(K)$ to the ring of "formal power series" $\widehat{A}(K)$.

Let $A\langle\langle K\rangle\rangle$ be an $A$-algebra generated in $\widehat{A}(K)$ by $A(K)$ and all the following series for all positive $\delta \in A$ and $a, b \in K$ :

$$(a^\delta - 1)/(a-1) = \Sigma_{0\leq\alpha<\delta} a^\alpha,$$

and

$$\Sigma_{0\leq\alpha<\delta} b^\alpha \frac{a^\alpha - 1}{a-1}.$$

# Analogy with Hall's-completions

In the Hall's $R$-completions of nilpotent groups $N$ one has to extend the ring $R$ to a binomial ring $R_{bin}$ adding all the binomial coefficients $\binom{a}{n}$ for $a \in R$ and $n \in \mathbb{N}$, so the functions that define the multiplication of normal forms (Malcev's coordinates) in $N \otimes_H R_{bin}$ are well-defined.

# $A$-metabelian groups

Define an $A$-metabelian exponential group $M$ by the following axioms:

- $M$ is metabelian $A$-exponential group.
- $M'$ is $A\langle\langle K\rangle\rangle$-module.
- For any $z, x \in M$ and $\delta \in A$,

$$[z, x^\delta] = [z, x]^{(a^\delta - 1)/(a-1)}.$$

- For any $z, x \in M$ and $\delta \in A$,

$$y^{-\delta} x^{-\delta} (xy)^\delta = [x, y]^{f(a,b)},$$

where

$$f(a, b) = [(a^\delta b^\delta - 1)/(ab - 1) + (1 - b^\delta)/(b - 1)]/(1 - a).$$