Quantisation ideals: multi-quantum systems and systems with non-deformation quantisation

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Outline

Classical vs Quantum Mechanics

Classical Mechanics	Quantum Mechanics
$\mathfrak{A}_0 = \mathbb{R}[p,q],\ pq - qp = 0,$	$\mathfrak{A}_\hbar = \mathbb{C}(\hbar)\langle \hat{p}, \hat{q}\rangle \big/ \langle \hat{p}\hat{q} - \hat{q}\hat{p} + i\hbar\rangle$
$\partial_t q = p, \quad \partial_t p = F(q) = -U'(q)$	$\partial_t \hat{q} = \hat{p}, \quad \partial_t \hat{p} = F(\hat{q})$
$H = \frac{1}{2}p^2 + U(q), \quad \partial_t(H) = 0,$	$\hat{H} = \frac{1}{2}\hat{p}^2 + U(\hat{q}), \quad \partial_t(\hat{H}) = 0,$
$\{q,p\} = 1,$	$[\hat{q},\hat{p}] = i\hbar,$
$\partial_t q = \{q,H\},\ \partial_t p = \{p,H\},$	$\partial_t \hat{q} = \frac{i}{\hbar}[\hat{H},\hat{q}],\ \partial_t \hat{p} = \frac{i}{\hbar}[\hat{H},\hat{p}],$
$\partial_t a = \{a,H\},$	$\partial_t \hat{a} = \frac{i}{\hbar}[H,\hat{a}].$

Real valued functions $p, q, H \longleftrightarrow$ Hermitian operators $\hat{p}, \hat{q}, \hat{H}$

Canonical and deformation quantisation

1. Canonical quantisation:

$$q_i \to \hat{q}_i = q_i, \qquad p_i \to \hat{p}_i = p_i - i\hbar\frac{\partial}{\partial q_i}, \qquad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}.$$

2. Deformation quantisation (deformation of the multiplication):

$$f \cdot g \longrightarrow f \star g = f \cdot g + \hbar B_1(f,g) + \hbar^2 B_2(f,g) + \cdots$$

$$f \star g - g \star f = i\hbar\{f,g\} + \mathcal{O}(\hbar^2).$$

$$i\{f,g\} = B_1(f,g) - B_1(g,f).$$

Issues:

- Canonical transformations – a choice of canonical variables.
- Associativity of the deformed non-commutative multiplication.
- Consistency of the algebra with the equations of motion for finite $\hbar$.
- Ordering of operators in the Hamiltonian and other observables.

Uncharted territories:

- Quantisation of systems admitting a multi-Hamiltonian structure.
- Can we define a non-deformation quantisation?

Free Associative Algebra $\mathfrak{A} = \mathbb{C}\langle q, p\rangle$:
As a linear space the algebra $\mathfrak{A}$ has a basis of monomials

$$Mon(\mathfrak{A}) = \{p^{i_1} q^{j_1} p^{i_2} q^{j_2} \cdots p^{i_m} q^{j_m} \mid i_k, j_k \in \mathbb{Z}_{\geq 0}\}.$$

The number of monomials of a degree $n = i_1 + j_1 + \cdots + i_m + j_m$ is 2^n.

A dynamical system on $\mathfrak{A}$ is a derivation of $\mathfrak{A} = \mathbb{C}\langle p, q\rangle$

$$\partial_t : \mathfrak{A} \mapsto \mathfrak{A}, \qquad \partial_t(a\,b) = \partial_t(a)\,b + a\,\partial_t(b), \quad \forall\ a, b \in \mathfrak{A}.$$

Derivation ∂_t is defined by its action on the generators:

$$\partial_t q = A(p, q), \qquad \partial_t p = B(p, q) \qquad A(p, q), B(p, q) \in \mathfrak{A}.$$

Example: Let $\partial_t q = p, \quad \partial_t p = 3q^2,$ (*). There are two types of "first integrals"

- $\hat{H} = p\cdot q - q\cdot p$
  $\hat{H}_t = p_t\cdot q + p\cdot q_t - q_t\cdot p - q\cdot p_t = 3q^2\cdot q + p\cdot p - p\cdot p - q\cdot 3q^2 = 0.$
- The Hamiltonian of the classical system $H_1 = \frac{1}{2}p^2 - q^3$ is not a constant of motion

$$\begin{aligned}\partial_t(H_1) &= \tfrac{1}{2}(\dot{p}p + p\dot{p}) - (\dot{q}q^2 + q\dot{q}q + q^2\dot{q})\\ &= \tfrac{1}{2}(pq^2 - 2qpq + q^2p)\\ &= \tfrac{1}{2}[pq, q] + \tfrac{1}{2}[q, qp] \neq 0, \quad \partial_t(H) \in \mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}].\end{aligned}$$

  The system admits infinitely many algebraically independent "first integrals" $H_k \in \mathfrak{A}/\mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}]$, such that $\partial_t(H_k) \in \mathrm{Span}_{\mathbb{C}}[\mathfrak{A}, \mathfrak{A}]$.

$$H_2 = pqpq - p^2q^2, \quad H_3 = pqpq^2 - p^2q^3 - H_1^2, \ldots$$

- Nonabelian Newton's equation (*) admits infinitely many higher symmetries (derivations commuting with ∂_t):

$$\begin{aligned}\partial_{t_5} q &= q^2p - 2qpq + pq^2,\\ \partial_{t_5} p &= -qp^2 + 2pqp - p^2q;\end{aligned}$$

$$\begin{aligned}\partial_{t_7} q &= 2pq^3 - p^3 + 2q^3p - q^2pq - qpq^2,\\ \partial_{t_7} p &= -2q^2p^2 + qpqp - 2qp^2q + pq^2p + pqpq - 2p^2q^2 + 6q^5; \cdots\end{aligned}$$

Problem of quantisation, quantisation ideals

Fact: Any finitely generated associative algebra can be realised as a quotient of a free algebra $\mathfrak{A}$ over an appropriate two sided ideal $\mathfrak{J}$.

In Algebraic Quantisation, the problem of quantisation of a free associative dynamical system (i.e. a derivation $\partial_t : \mathfrak{A} \mapsto \mathfrak{A}$) can be formulated as:

To find a two sided ideal $\mathfrak{J} \subset \mathfrak{A}$ such that

A. $\partial_t(\mathfrak{J}) \subseteq \mathfrak{J} \quad \Leftrightarrow$ the derivation ∂_t induces a derivation of the quotient algebra $\mathfrak{A}/\mathfrak{J}$.

B. The quotient algebra $\mathfrak{A}/\mathfrak{J}$ has an additive basis of normally ordered monomials. In other words, we know how to change the order of any two variables.

An ideal $\mathfrak{J}$ satisfying the conditions A, B is called a quantisation ideal and the corresponding quotient algebra $\mathfrak{A}/\mathfrak{J}$ a quantised algebra.

Triangular ideals

Let $\mathfrak{A} = \mathbb{C}\langle x_1, \ldots, x_n\rangle$ be a free associative $\mathbb{C}$ algebra equipped with a lexicographic ordering of variables $x_i \succ x_j$ and monomials $x_j x_i \succ x_i x_j;\ i > j$. It has the additive basis of monomials

$$Mon(\mathfrak{A}) = \left\{ x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_m}^{\alpha_m} \,|\, n \geq i_1, i_2, \ldots, i_m \geq 1,\ \alpha_k \geq 0 \right\}$$

Let $\mathfrak{J} = \langle f_{i,j} \,|\, 1 \leq i < j \leq n\rangle$ be an ideal in $\mathfrak{A}$ where

$$f_{i,j} = x_i x_j - \omega_{i,j} x_j x_i + g_{i,j}, \quad g_{i,j} \in \mathfrak{A},\ Lm(g_{i,j}) \prec x_j x_i \prec x_i x_j, \quad \omega_{i,j} \neq 0.$$

$$\omega_{i,k}\omega_{j,k} g_{i,j} x_k - x_k g_{i,j} + \omega_{j,k} x_j g_{i,k} - \omega_{i,j} g_{i,k} x_j + g_{j,k} x_i - \omega_{i,j}\omega_{i,k} x_i g_{j,k} = 0.$$

Then in the quotient $\mathfrak{A}/\mathfrak{J}$ there is a monomial basis of ordered monomials

$$Mon(\mathfrak{A}/\mathfrak{J}) = \left\{ x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_m}^{\alpha_m} \,|\, n \geq i_1 > i_2 > \ldots > i_m \geq 1,\ \alpha_k \geq 0 \right\}$$

Algebra $\mathfrak{A}/\mathfrak{J}$ has Poincaré–Birkhoff–Witt basis of normally ordered monomials (V.Levandovskyy 2005) and thus satisfies the condition (B).

Sklyanin's quadratic algebra is PBW, but the ideal is not triangular.

Graded Superalgebras can be fit in the above scheme, but they are not of PBW type.

Problem: To give a classification of PBW ideals.

Volterra system on the free algebra $\mathfrak{A} = \mathbb{C}\langle u_n;\ n \in \mathbb{Z}\rangle$.

$$\partial_{t_1} u_n = u_{n+1} u_n - u_n u_{n-1}, \qquad n \in \mathbb{Z}. \tag{1}$$

Periodic closures of the chains $u_{k+M} = u_k$ with period M result in nonabelian systems on $\mathfrak{A}^M = \mathbb{C}\langle u_1, \dots, u_M\rangle$.

Let $M = 3$:

$$\begin{aligned} u_{1,t_2} &= u_2 u_1 - u_1 u_3, \\ u_{2,t_2} &= u_3 u_2 - u_2 u_1, \\ u_{3,t_2} &= u_1 u_3 - u_3 u_2 \end{aligned}$$

It has an obvious constant of motion $H = u_1 + u_2 + u_3$.

It has infinitely many commuting symmetries:

$$\begin{aligned} u_{1,t_3} \quad &= u_1^2 u_3 + u_1 u_3 u_2 + u_1 u_3^2 - u_2 u_1^2 - u_2^2 u_1 - u_3 u_2 u_1, \\ u_{1,t_4} \quad &= u_1^3 u_3 + u_1^2 u_3 u_2 + u_1^2 u_3^2 + u_1 u_2 u_1 u_3 + u_1 u_3 u_1 u_3 + u_1 u_3 u_2^2 \\ &+ u_1 u_3 u_2 u_3 + u_1 u_3^2 u_2 + u_1 u_3^3 - u_2 u_1^3 - u_2 u_1 u_2 u_1 - u_2 u_1 u_3 u_1 \\ &- u_2^2 u_1^2 - u_2^3 u_1 - u_2 u_3 u_2 u_1 - u_3 u_2 u_1^2 - u_3 u_2^2 u_1 - u_3^2 u_2 u_1, \dots \ . \end{aligned}$$

Quantisation of the periodic Volterra chains

For periodic Volterra systems with period M we consider triangular ideals

$$\mathfrak{I}_M = \langle u_q u_p - \omega_{p,q} u_p u_q - \sum_{r=1}^{M} \sigma_{p,q}^r u_r - \eta_{p,q}\,;\; 1 \le q < p \le M, \quad \omega_{p,q} \neq 0 \rangle.$$

Proposition

Nonabelian periodical Volterra chain with period M admits $\mathfrak{I}_M$–quantisation iff

$$\begin{array}{ll} M = 3: & u_n u_{n+1} = \alpha u_{n+1} u_n + \beta(u + u_1 + u_2) + \eta, \quad n \in \mathbb{Z}_3; \\ M = 4: & u_1 u_2 = \alpha u_2 u_1 + \beta u_2 + \gamma u_1 - \beta\gamma, \\ & u_1 u_3 = u_3 u_1 - \beta u_2 + \beta u_4, \\ & u_4 u_1 = \alpha u_1 u_4 + \beta u_4 + \gamma u_1 - \beta\gamma, \\ & u_2 u_3 = \alpha u_3 u_2 + \beta u_2 + \gamma u_3 - \beta\gamma, \\ & u_2 u_4 = u_4 u_2 - \gamma u_3 + \gamma u_1, \\ & u_3 u_4 = \alpha u_4 u_3 + \beta u_4 + \gamma u_3 - \beta\gamma; \\ M \geq 5: & u_{n+1} u_n = \alpha u_n u_{n+1}, \\ & u_n u_m = u_m u_n, \quad |n - m| > 1, \; n, m \in \mathbb{Z}_M, \end{array}$$

where the constants $\alpha, \beta, \gamma, \eta \in \mathbb{C}$, $\alpha \neq 0$ are arbitrary.
Infinite Volterra hierarchy admits quantisation

$$u_{n+1} u_n = \alpha u_n u_{n+1}, \quad u_n u_m = u_m u_n, \quad |n - m| > 1, \; n, m \in \mathbb{Z}.$$

Classical commutative case:

$$\begin{aligned}
u_{1,t} &= u_2 u_1 - u_1 u_3, & H_1 &= u_1 + u_2 + u_3,\\
u_{2,t} &= u_3 u_2 - u_2 u_1, & H_2 &= u_1 u_2 u_3,\\
u_{3,t} &= u_1 u_3 - u_3 u_2, & \{u_{n+1}, u_n\} &= \mu u_{n+1} u_n + \nu, \quad n \in \mathbb{Z}_3,
\end{aligned}$$

$$\begin{aligned}
\{u_{n+1}, u_n\}_\mu &= u_{n+1} u_n, & \{u_{n+1}, u_n\}_\nu &= -1,\\
\{u_k, H_2\}_\mu &= 0 & \{u_k, H_1\}_\nu &= 0\\
u_{k,t} &= \{u_k, H_1\}_\mu, & u_{k,t} &= \{u_k, H_2\}_\nu, .
\end{aligned}$$

Let $\mathfrak{J}_{\theta,\hbar} = \langle q^{-1} u_n u_{n+1} - q u_{n+1} u_n - \mathrm{i}\theta\,;\ n \in \mathbb{Z}_3,\ q = e^{\mathrm{i}\hbar} \rangle$.
Algebra $\mathfrak{A}^{(\theta,\hbar)} = \mathfrak{A}_3 \diagup \mathfrak{J}_{\theta,\hbar}$ has a central element

$$\mathcal{H}(\theta,\hbar) = \sin(\hbar) H_2 + \theta(2 + \cos(2\hbar)) H_1,$$

where the self-adjoint elements

$$\begin{aligned} H_1 &= u_1 + u_2 + u_3, \\ H_2 &= \sum_{\sigma \in \Sigma_3} u_{\sigma(1)} u_{\sigma(2)} u_{\sigma(3)} \\ &= 3(q^2 + 1) u_3 u_2 u_1 + \mathrm{i}\theta \left((2q + q^{-1})(u_1 + u_3) - (q + 2q^{-1}) u_2 \right) \end{aligned}$$

are first integrals for the quantum Volterra system

$$(u_n)_{t_1} = q(u_{n+1} u_n - u_n u_{n-1}), \qquad n \in \mathbb{Z}_3.$$

Moreover, it can be represented in the Heisenberg form

$$(u_n)_{t_1} = \frac{\mathrm{i}}{2 \sin \hbar} [H_1, u_n] = -\frac{\mathrm{i}}{2\theta(2 + \cos(2\hbar))} [H_2, u_n].$$

With two quotient algebras $\mathfrak{A}^{(\theta,0)}$ and $\mathfrak{A}^{(0,\hbar)}$ we associate the bi-quantum structure (a quantum deformation of the bi-Hamiltonian structure) as follows:

parameters	$\theta \neq 0,\ q = 1$	$\theta = 0,\ q = e^{\mathrm{i}\hbar}$
commutation relations	$[u_n, u_{n+1}] = \mathrm{i}\theta$	$u_n u_{n+1} = q^2 u_{n+1} u_n, \quad n \in \mathbb{Z}_3$
central element	$H_1 = u_1 + u_2 + u_3$	$H_2 = q u_3 u_2 u_1$
the Heisenberg form	$(u_n)_{t_1} = -\frac{\mathrm{i}}{\theta}[H_2, u_n]$	$(u_n)_{t_1} = \frac{\mathrm{i}}{2\sin\hbar}[H_1, u_n]$

Quantisation of the periodic Volterra cubic sysmmetry

The M–periodic cubic Volterra's symmetry:

$$\partial_{t_2} u_n = u_{n+2}u_{n+1}u_n + u_{n+1}^2 u_n + u_{n+1}u_n^2 - u_n^2 u_{n-1} - u_n u_{n-1}^2 - u_n u_{n-1} u_{n-2}, \quad n \in \mathbb{Z}_M$$

- In the case $M = 3$ the quantisation ideal is generated by relations:

$$u_n u_{n+1} = \alpha u_{n+1} u_n + \beta(u_1 + u_2 + u_3) + \eta, \quad n \in \mathbb{Z}_3$$

- For odd $M \geq 5$ the quantisation ideal is generated by relations

$$u_{n+1}u_n = \alpha u_n u_{n+1}, \quad u_n u_m = u_m u_n, \quad |n-m| > 1, \ n, m \in \mathbb{Z}_M$$

- For even $M \geq 6$ and for the infinite system there are two quantisations

  1. $u_{n+1}u_n = \alpha u_n u_{n+1}, \quad u_n u_m = u_m u_n, \quad |n-m| > 1, \ n, m \in \mathbb{Z}_M;$
  2. $u_n u_{n+1} = (-1)^n \omega u_{n+1} u_n, \quad u_n u_m + u_m u_n = 0$ if $|n-m| \geqslant 2, \quad n, m \in \mathbb{Z}_M$

The $M = 4$ periodical reduction of the cubic Volterra symmetry ($n \in \mathbb{Z}_4$):

$$\partial_{t_2} u_n = u_{n+2}u_{n+1}u_n + u_{n+1}^2 u_n + u_{n+1}u_n^2 - u_n^2 u_{n+3} - u_n u_{n+3} u_{n+2} - u_n u_{n+3}^2$$

Theorem. The above nonabelian system admits a quantisation with the ideal of the form

$$\mathfrak{J} = \langle u_i u_j - \omega_{i,j} u_j u_i \, ; \, 1 \leqslant i < j \leqslant 4 \rangle$$

iff the six constants $\omega_{i,j} \neq 0$ take values as in one of four cases:

	$\omega_{1,2}$	$\omega_{1,3}$	$\omega_{2,3}$	$\omega_{1,4}$	$\omega_{2,4}$	$\omega_{3,4}$
(a) :	ω,	1,	ω,	ω^{-1},	1,	ω;
(b) :	ω,	-1,	$-\omega$,	$-\omega^{-1}$,	-1,	ω;
(c) :	$-\omega$,	-1,	ω,	$-\omega^{-1}$,	1,	ω;
(d) :	$-\omega$,	1,	$-\omega$,	ω^{-1},	-1,	ω.

Moreover, in each of four cases the quantised system is a super-integrable quantum system.

Non-deformation quantisation

The quantisation (b) ideal $\mathfrak{J}_b$ is generated by:

$$\begin{aligned} uu_1 &= \omega u_1 u, & uu_2 &= -u_2 u, & uu_3 &= -\omega u_3 u, \\ u_1 u_2 &= -\omega u_2 u_1, & u_1 u_3 &= -u_3 u_1, & u_2 u_3 &= \omega u_3 u_2. \end{aligned}$$

The algebra $\mathfrak{A}_4/\mathfrak{J}_b$ has three central elements ($\hat{H}^2 = \hat{H}_1 \hat{H}_2$):

$$\hat{H} = u_3 u_2 u_1 u, \qquad \hat{H}_1 = u_3^2 u_1^2, \qquad \hat{H}_2 = u_2^2 u^2.$$

The dynamical system on $\mathfrak{A}_4/\mathfrak{J}_b$ admits three first integrals

$$H = u + u_1 + u_2 + u_3, \qquad H_1 = u_3 u_1, \qquad H_2 = u_2 u.$$

$$H^2 = \sum_{n \in \mathbb{Z}_4} (u_n^2 + u_n u_{n+1} + u_{n+1} u_n), \qquad [H^2, H_1] = [H^2, H_2] = [H_1, H_2] = 0.$$

On $\mathfrak{A}_4/\mathfrak{J}_b$ the quantum Volterra system can be written as

$$u_{n,t_3} = \frac{1}{\omega^2 - 1}[H^2, u_n]$$

$$= u_{n+2} u_{n+1} u_n + u_{n+1}^2 u_n + u_{n+1} u_n^2 - u_n^2 u_{n+3} - u_n u_{n+3} u_{n+2} - u_n u_{n+3}^2, \qquad n \in \mathbb{Z}_4.$$

Let $\omega = 1 + \nu$ and $\nu \to 0$, then $\mathfrak{A}/\mathfrak{J}_b \to \mathfrak{A}_0 = \mathfrak{A}/\mathfrak{J}_b^0$

in $\mathfrak{A}/\mathfrak{J}_b$: $ab = (a, b)_0 + \nu(a, b)_1 + \cdots$, $\quad [a, b] = \{a, b\}_0 + \nu\{a, b\}_1 + \nu^2\{a, b\}_2 + \cdots$

$\mathfrak{J}_b \to \mathfrak{J}_b^0 = \langle uu_1 - u_1 u, uu_2 + u_2 u, uu_3 + u_3 u, u_1 u_2 + u_2 u_1, u_1 u_3 + u_3 u_1, u_2 u_3 - u_3 u_2 \rangle$

$$H^2 = \mathcal{H}_0 + \nu \mathcal{H}_1, \quad \mathcal{H}_0 \in Z(\mathfrak{A}_0), \quad \mathcal{H}_1 \in \mathfrak{A}_0$$

Let

$$\mathbf{H} = (\mathcal{H}_0, \mathcal{H}_1 + Z(\mathfrak{A}_0)) \in \Pi(\mathfrak{A}_0) := Z(\mathfrak{A}_0) \oplus (\mathfrak{A}_0 / Z(\mathfrak{A}_0))$$

then

$$\frac{1}{\nu}[H^2, a] \to \partial_{\mathbf{H}}(a) = \{\mathcal{H}_0, a\} + [\mathcal{H}_1, a], \quad \partial_{\mathbf{H}} : \mathfrak{A}_0 \mapsto \mathfrak{A}_0.$$

Moreover

$$\partial_{\mathbf{H}}(\partial_{\mathbf{G}}(a)) - \partial_{\mathbf{G}}(\partial_{\mathbf{H}}(a)) = \partial_{\{\mathbf{H},\mathbf{G}\}}(a), \qquad \mathbf{H}, \mathbf{G} \in \Pi(\mathfrak{A}_0).$$

$\Pi(\mathfrak{A}_0)$ is a commutative Poisson algebra. Explicit formulas for $\times$ and the Poisson brackets are given by: for $(a, \overline{a_1}), (b, \overline{b_1}) \in \Pi(\mathfrak{A}_0)$

$$(a, \overline{a_1}) \times (b, \overline{b_1}) = \left(ab, \overline{ab_1 + a_1 b + (a, b)_1}\right) ,$$

$$\left\{(a, \overline{a_1}), (b, \overline{b_1})\right\} = \left(\{a, b\}_1, \overline{\{a, b\}_2 + \{a_1, b\}_1 + \{a, b_1\}_1 + [a_1, b_1]}\right) .$$

Bi-Hamiltonan structure of the Toda hierarchy

Toda lattice:

$$\frac{d^2 x_k}{dt_1^2} = e^{x_{k+1}-x_k} - e^{x_k - x_{k-1}}, \qquad k \in \mathbb{Z} \text{ (or } \ k \in \mathbb{Z}_N)$$

In variables

$$b_k = (x_k)_{t_1}, \qquad a_k = e^{x_{k+1}-x_k}, \qquad H_1 = \sum b_k,$$

$$(b_k)_{t_1} = a_k - a_{k-1}, \qquad (a_k)_{t_1} = b_{k+1}a_k - a_k b_k, \qquad H_2 = \sum \tfrac{1}{2} b_k^2 + a_k,$$

$$\{a_k, b_k\}_1 = a_k, \qquad \{b_{k+1}, a_k\}_1 = a_k, \qquad \{\bullet, H_1\}_1 = 0$$

$$\{a_k, b_k\}_2 = b_k a_k, \qquad \{b_{k+1}, a_k\}_2 = a_k b_{k+1},$$

$$\{b_{k+1}, b_k\}_2 = a_k, \qquad \{a_{k+1}, a_k\}_2 = a_k a_{k+1};$$

$$(b_k)_{t_1} = \{b_k, H_2\}_1 = \{b_k, H_1\}_2 = a_k - a_{k-1},$$

$$(a_k)_{t_1} = \{a_k, H_2\}_1 = \{a_k, H_1\}_2 = b_{k+1}a_k - a_k b_k$$

$$(b_k)_{t_2} = \{b_k, H_3\}_1 = \{b_k, H_2\}_2 = a_k b_k - b_k a_{k-1} - a_{k-1} b_{k-1} + b_{k+1} a_k$$

$$(a_k)_{t_2} = \{a_k, H_3\}_1 = \{a_k, H_2\}_2 = a_{k+1} a_k - a_k a_{k-1} + b_{k+1}^2 a_k - a_k b_k^2$$

$$\{\bullet, \bullet\}_{\lambda,\mu} = \lambda\{\bullet, \bullet\}_1 + \mu\{\bullet, \bullet\}_2.$$

In the quantum case we assume variables a, b to be self adjoint, then

$$\partial_t a = e^{i\hbar}(b_1 a - ab), \qquad b_t = a - a_{-1}.$$

and the ∂_t–stable, $\mathcal{S}$ and $\dagger$ invariant ideal $\mathfrak{I}_{\theta,\hbar}$ is generated by polynomials:

$$\begin{array}{lll} b_n b_{n+1} - b_{n+1} b_n - 2i\sin(\hbar)\, a_n, & b_n b_m - b_m b_n, & |n-m| \neq 1, \\ e^{-i\hbar} a_n a_{n+1} - e^{i\hbar} a_{n+1} a_n, & a_n a_m - a_m a_n, & |n-m| \neq 1, \\ e^{-i\hbar} b_n a_n - e^{i\hbar} a_n b_n - i\theta a_n, & a_n b_m - b_m a_n, & m-n \neq 0, 1, \\ e^{-i\hbar} a_n b_{n+1} - e^{i\hbar} b_{n+1} a_n - i\theta a_n. & & \end{array} \qquad (2)$$

It depends on two real parameters $\theta, \hbar$.
Commuting self-adjoint first integrals:

$$H_1 = \sum_{n\in\mathbb{Z}} b_n,$$

$$H_2 = \frac{1}{2}\sum_{n\in\mathbb{Z}} (b_n^2 + 2a_n + [b_{n+1}, b_n]),$$

$$H_3 = \frac{1}{3}\sum_{n\in\mathbb{Z}} (b_n^3 + 2a_n(b_n + b_{n+1}) + (b_n + b_{n+1})a_n + [b_n^2, b_{n+1}] + [b_n b_{n+1}, b_{n+1}]).$$

Bi-quantisation of the Toda system

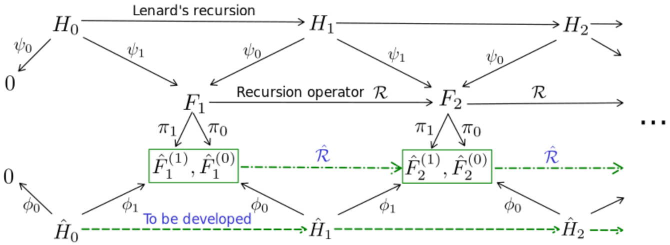
Classical 1	Quantum $\mathfrak{I}_{\theta,0}$
$\{a_k, b_k\}_1 = a_k,$	$[a_k, b_k]_\theta = i\theta a_k,$
$\{b_{k+1}, a_k\}_1 = a_k,$	$[b_{k+1}, a_k]_\theta = i\theta a_k,$
Classical 2	**Quantum $\mathfrak{I}_{0,\hbar}$**
$\{a_k, b_k\}_2 = b_k a_k,$	$b_k a_k = e^{2i\hbar} a_k b_k,$
$\{b_{k+1}, a_k\}_2 = a_k b_{k+1},$	$a_k b_{k+1} = e^{2i\hbar} b_{k+1} a_k,$
$\{b_{k+1}, b_k\}_2 = a_k,$	$[b_k, b_{k+1}] = 2i\sin(\hbar) a_k,$
$\{a_{k+1}, a_k\}_2 = a_k a_{k+1},$	$a_k a_{k+1} = e^{2i\hbar} a_{k+1} a_k.$

$$\{H_1, \bullet\}_1 = 0 \qquad [H_1, \bullet]_\theta = 0,$$

$$\partial_{t_1}(\bullet) = \frac{i}{2\sin\hbar}[H_1, \bullet]_\hbar, \qquad \partial_{t_1}(\bullet) = \frac{i}{\theta}[H_2, \bullet]_\theta,$$

$$\partial_{t_2}(\bullet) = \frac{i}{\sin(2\hbar)}[H_2, \bullet]_\hbar, \qquad \partial_{t_2}(\bullet) = \frac{i}{\theta}[H_3, \bullet]_\theta, \cdots .$$

Multi quantum structure and quantum analogue of the Lenard-Magri scheme?



What about quantisation of Toda on algebras $A_n^{(2)}$, $B_n^{(1)}$, $D_n^{(2)}$, $D_4^{(3)}$, ...?

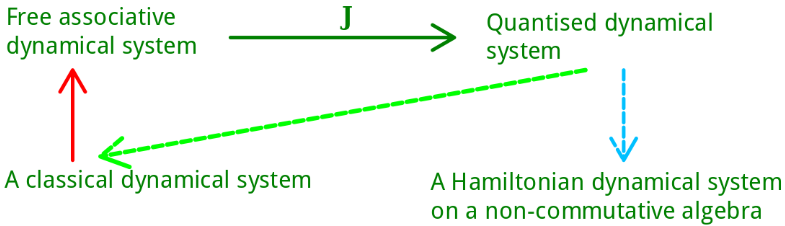
$$\partial_{t_2} a = e^{i\hbar}(a_1 a - a a_{-1}) + b_1 a b_1 - bab, \quad \partial_{t_2} b = e^{i\hbar}(ab - a_{-1} b_{-1} + b_1 a - b a_{-1})$$

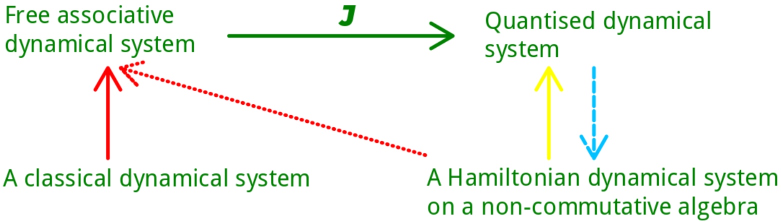
Proposition

The ideal $\hat{\mathfrak{J}}$ generated by the following set of commutation relations

$$\begin{aligned} b_n b_{n+1} + b_{n+1} b_n &= 2\cos(\hbar)\, a_n, & b_n b_m + b_m b_n = 0, \quad & |n-m| \neq 1, \\ e^{-i\hbar} a_n a_{n+1} &= e^{i\hbar} a_{n+1} a_n, & a_n a_m - a_m a_n = 0, \quad & |n-m| \neq 1, \\ e^{-i\hbar} b_n a_n &= e^{i\hbar} a_n b_n, & a_n b_m - b_m a_n = 0, \quad & m-n \neq 0, 1, \\ e^{-i\hbar} a_n b_{n+1} &= e^{i\hbar} b_{n+1} a_n. & & \end{aligned} \tag{3}$$

is stable with respect to the derivation ∂_{t_2}.


Free associative dynamical system
J
Quantised dynamical system
A classical dynamical system
A Hamiltonian dynamical system on a non-commutative algebra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