The rational subset membership problem: a survey and new results

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Rational sets

In computer science, more precisely in automata theory, a rational set of a monoid is an element of the minimal class of subsets of this monoid that contains all finite subsets and is closed under union, product and Kleene star. Rational sets are useful in automata theory, formal languages and algebra.
A rational set generalizes the notion of rational (regular) language (understood as defined by regular expressions) to monoids (in particular, groups) that are not necessarily free.
Below we consider rational sets only over groups.

Rational subsets of a group

Denote by $\mathrm{Rat}(G)$ the class of rational subsets of the group G. By definition, $\mathrm{Rat}(G)$ is the smallest class that contains all finite subsets of G and that is closed with respect to the following operations:

• union: $R_1, R_2 \in \mathrm{Rat}(G) \ \Rightarrow \ R_1 \cup R_2 \in \mathrm{Rat}(G)$;

• product:
$R_1, R_2 \in \mathrm{Rat}(G) \ \Rightarrow \ R_1 \cdot R_2 = \{r_1 r_2 \colon r_1 \in R_1,\ r_2 \in R_2\} \in \mathrm{Rat}(G)$;

• taking the monoid generated by a set:
$R \in \mathrm{Rat}(G) \ \Rightarrow \ R^* = \bigcup_{i=1}^{\infty} R^i \cup \{1\} \in \mathrm{Rat}(G)$.

Examples of rational subsets are finitely generated (and only finitely generated) subgroups and finitely generated submonoids. In the general case, the family $\mathrm{Rat}(G)$ is not closed under the operations of intersection and complement.

Finite automata

A finite automaton defined over a group G is a finite directed graph with

1. Edges labelled by elements of G;
2. An initial vertex;
3. A set of terminal vertices.

By a path in an automaton we mean a directed path. The label of a path is the product in order of the labels of its edges; in particular the label of an empty path, i.e., a path of length zero from a vertex to itself, is the unit element.
By definition, the set accepted by an automaton is the collection of labels of successful paths. A successful path is one from the initial vertex to a terminal vertex.

Kleene's theorem

Theorem

Let G be a group. A subset R of G is rational if and only if it is accepted by an automaton over G.

Well-known positive results

We present some well-known results on the rational subset membership problem for a group.
Positive results:

• M. Benois (1969). The rational subset membership problem is decidable for free groups.

• S. Eilenberg, M. P. Sch́utzenberger (1969). The rational subset membership problem is decidable for abelian groups.

• Z. Grunschlag (1999). The decidability of the rational subset membership problem is preserved under finite extensions of groups.

• M. Yu. Nedbay (2000). The decidability of the rational subset membership problem is preserved under free products of groups.

bullet T. Colcombet, J. Ouaknine, P. Semukhin, and J. Worrell proved the decidability of the submonoid membership problem for the Heisenberg group.

Well-known negative results

Negative results:

•[1] V. A. Roman'kov (1999). For any $l \geq 2$ and sufficiently large r the rational subset membership problem for a free nilpotent group $N_{r,l}$ is undecidable.
•[2] M. Lohrey, B. Steinberg (2011). The free metabelian group M_r of rank $r \geq 2$ contains a finitely generated submonoid for which the membership problem is undecidable.
The proofs in [1] are based on the undecidability of Hilbert's tenth problem, and in [2] on the undecidability of the problem of the existence of a combinatorial tiling.

Surveys

For other numerous results on the rational subset membership problem, see surveys:

M. Lohrey, The rational subset membership problem for groups: a survey, Groups St. Andrews 2013, London Math. Soc. Lecture Note Ser., Vol. 422, Cambridge Univ. Press, Cambridge, 2015, 368–389.

L. Bartholdi, Pedro V. Silva, Rational subsets of groups. Published in Handbook of Automata, 2010.

M. Kambites, Pedro V. Silva, B. Steinberg. On the rational subset problem for groups, Journal of Algebra, Vol. 309, Iss. 2, 15 March 2007, 622–639.

The submonoid membership problem for finitely generated nilpotent groups: general case

The submonoid membership problem for finitely generated nilpotent groups has attracted the attention of a number of researchers in recent years. This is the problem of the existence of an algorithm that determines, given an arbitrary element g and a finitely generated submonoid M of a group G, whether g belongs to M. This question was asked by many authors from the beginning of 21 Century. It was explicitly posed by M. Lohrey in his observing paper at 2013 as Open problem 24).
By, the question was raised about the decidability of the submonoid membership problem for a direct power of the Heisenberg group.

The submonoid membership problem for finitely generated nilpotent groups: a direct power of the Heisenberg group

T. Colcombet, J. Ouaknine, P. Semukhin, and J. Worrell posed the question about the decidability of the submonoid membership problem for a direct power of the Heisenberg group.

Motivation for the submonoid membership problem: natural class

Firstly, submonoids are a classical subclass of rational sets of a group, a natural generalization of subgroups. The subgroup membership problem for finitely generated nilpotent groups is solvable by Maltsev's theorem. At the same time, the submonoid membership problem, a natural, more general subclass of rational sets, remained open in these groups for a long time.

Motivation for the submonoid membership problem: noncommutative version of the classical problem of integer linear programming

Secondly, the submonoid membership problem for a noncommutative group is currently considered as a transfer of the classical problem of integer linear programming, where the membership problem in a submonoid of a free abelian group appears, to a noncommutative platform. The problem is most interesting for nilpotent groups, as the closest class to abelian groups.

Motivation for the submonoid membership problem: a version of the integer linear programming problem

For example, the submonoid membership problem for the free abelian group $A_r \simeq \mathbb{Z}^r$ is related to the following integer linear programming problem: for a given integer matrix A of size $m \times r$ and the vector $b \in \mathbb{Z}^r$ determine whether there exists a solution $x \in \mathbb{N}^m$ of the equation $xA = b$. In the group-theoretic language, this is the membership problem in a submonoid of the group A_r generated by the rows of the matrix A. It is known that this version of the integer linear programming problem belongs to the class of NP-complete problems.

New results on the submonoid membership problem: equivalence of the Diophantine problem and the submonoid membership problem for free nilpotent groups

Let

$$D(\zeta_1, \ldots, \zeta_t) = \upsilon, \ \upsilon \in \mathbb{Z}, \qquad (1)$$

be an arbitrary Diophantine equation.

Theorem 1. For any Diophantine equation (1), there exists a sufficiently large rank r, and a finitely generated submonoid M of the free nilpotent group $N_{r,2}$, and an element $g(\upsilon) \in N_{r,2}$, whose can be effectively constructed, such that the equation (1) is solvable in integers if and only if $g(\upsilon)$ belongs to M.

We can change $N_{r,2}$ to $N_{r,l}$ for any $l \geq 2$.

New results on the submonoid membership problem: free nilpotent groups

The nondecidability of the tenth Hilbert problem and theorem 1 imply

Theorem 2. There exists a finitely generated submonoid M of a free nilpotent group $N_{r,l}$, $l \geq 2$, of sufficiently large rank r, the membership problem into which is algorithmically undecidable.

New results on the submonoid membership problem: equivalence of the Diophantine problem and the submonoid membership problem for direct powers of the Heisenberg group

Theorem 3. For any Diophantine equation (1), there exists a direct power $\mathbb{H}^n$ of the Heisenberg group $\mathbb{H}$, a finitely generated submonoid M in the group $\mathbb{H}^n$ and an element $g(v) \in \mathbb{H}^n$ such that the equation (1) is solvable in integers if and only if $g(v)$ belongs to M. The exponent n, the element $g(v)$, and the finite set of generators of the submonoid M are effectively determined. The submonoid M depends only on the Diophantine polynomial D on the left side (1).

New results on the submonoid membership problem: direct powers of the Heisenberg groups

The nondecidability of the tenth Hilbert problem and theorem 3 imply

Theorem 4. There exists a sufficiently large n, and a finitely generated submonoid M of the direct product $\mathbb{H}^n$, the membership problem into which is algorithmically undecidable.

The product of subgroups membership problem for groups

The product of subgroups membership problem for a finitely generated group G is the decision problem, where for a given two or more of finitely generated subgroups K_i, $i = 1, \ldots, m$, of G and a group element g it is asked whether

$$g \in \prod_{i=1}^{m} K_i.$$

Motivation for the product of subgroups membership problem: connection with the cosets intersection problem

The study of this problem was stimulated by the following question of E. Ventura Capell: Is the coset intersection problem – given $u, u', v_1, \dots v_r, v'_1, \dots, v'_s \in G$, decide whether coset intersections $u \cdot \mathrm{gp}(v_1, \dots, v_r)$ and $u' \cdot \mathrm{gp}(v'_1, \dots, v'_s)$ empty or not, decidable for metabelian groups?
In other words, does an element $g = u^{-1}u'$ belongs to the product KL where $K = \mathrm{gp}(u_1, \dots, u_r)$, $L = \mathrm{gp}(u'_1, \dots, u'_r)$?

The coset intersection problem

Calculations related to the intersections of cosets, in turn, play a key role in the famous E. Luks and L. Babai algorithms for determining graph isomorphisms.

New results on the product of subgroups membership problem for nilpotent groups: decidability

Theorem 5. Let K and L be a pair of finitely generated subgroups of a nilpotent group N of class two. Then the product membership problem in $F = KL$ is decidable for N.
Thus, we answer the Ventura question for any finitely generated nilpotent group of class two (which is metabelian) positively.

New results on the product of subgroups membership problem for nilpotent groups: nondecidability

Theorem 6. For any Diophantine equation (1), there exists a direct power $N = \mathbb{H}^n$ of the Heisenberg group $\mathbb{H}$, a four of finitely generated abelian subgroups $K_i, i = 1, \ldots, K_4$ of the group N and an element $g(\xi) \in N$ such that the equation (1) is solvable in integers if and only if $g(\xi)$ belongs to the product $K = \prod_{i=1}^{4} K_i$. The parameter n, the element $g(\xi)$, and the finite sets of generators of the subgroups K_i are effectively determined by (1). The subgroups K_i depend only on the Diophantine polynomial D on the left side of the equation (1).

New results on the product of subgroups membership problem for nilpotent groups: nondecidability

From the undecidability of Hilbert's 10th problem and theorem 6, it follows that the four subgroups membership problem in the class of finite direct powers of the Heisenberg group is undecidable.

Theorem 7. For sufficiently large $n \in \mathbb{N}$, the direct power $\mathbb{H}^n$ of the Heisenberg group $\mathbb{H}$ contains a product of four finitely generated subgroups with an unsolvable membership problem.

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V.A. Roman'kov. Unsolvability of the submonoid membership problem for a free nilpotent group of class $l \geq 2$ of a sufficiently large rank. Izvestiya Math. 47:4 (2023), 166–185.
V.A. Roman'kov. Undecidability of the submonoid membership problem for a sufficiently large finite direct power of the Heisenberg group. Siberian Electronic Mathematical Reports. 20:1 (2023), 293–305.
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Thanks

Thank you for your attention!