Semigroup equations and B. Plotkin‘s problem

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Aims and scope

Our research is related to universal algebraic geometry (UAG). See the papers, books and surveys by B.Plotkin, V. Remeslennikov, A.Miasnikov, E.Daniyarova and other researchers.
We study equations over semigroups and partially solve Plotkin's problem for wreath products of semigroups.

Languages, terms and atomic formulas. Examples

Let $\mathcal{L} = \{\cdot\}$ be a semigroup language. The examples of $\mathcal{L}$-terms are the following: x_1x_2, x_1x_1, $x_1x_2x_3x_3x_2x_1$, $\ldots$

An $\mathcal{L}$-atomic formula ($\mathcal{L}$-equation) is an equality of two $\mathcal{L}$-terms. The examples of $\mathcal{L}$-equations $x_1x_2 = x_2x_1$, $x_1x_1 = x_1$, $x_1x_2x_3 = x_4x_4x_4$, $\ldots$

Systems of equations

An arbitrary set of $\mathcal{L}$-equations ($\mathcal{L}$-atomic formulas) is called a system.

NB: we consider system of equations which depend on at most finite number of variables.

The solution set of a system of equations $\mathbf{S}$ over a semigroup $\mathcal{A}$ is denoted by $\mathrm{V}_{\mathcal{A}}(\mathbf{S})$.

An set $Y \subseteq \mathcal{A}^n$ is algebraic over $\mathcal{A}$ if there exists a system of $\mathcal{L}$-equations with the solution set Y.

Noetherian property (the main type of compactness)

An $\mathcal{L}$-algebra $\mathcal{A}$ is equationally Noetherian if any infinite $\mathcal{L}$-system $\mathbf{S}$ is equivalent over $\mathcal{A}$ to some finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$.

Equivalently, $\mathcal{A}$ is equationally Noetherian iff there is not any infinite chain

$$Y_1 \supset Y_2 \supset \ldots \supset Y_n \supset \ldots$$

of algebraic sets over $\mathcal{A}$.

An $\mathcal{L}$-algebra $\mathcal{A}$ is $\mathbf{q}_\omega$-compact if for any infinite $\mathcal{L}$-system $\mathbf{S}$ and an equation $t(X) = s(X)$ such that

$$\mathrm{V}_{\mathcal{A}}(\mathbf{S}) \subseteq \mathrm{V}_{\mathcal{A}}(t(X) = s(X))$$

there exists a finite subsystem $\mathbf{S}' \subseteq \mathbf{S}$ with

$$\mathrm{V}_{\mathcal{A}}(\mathbf{S}') \subseteq \mathrm{V}_{\mathcal{A}}(t(X) = s(X))$$

Any equationally Noetherian $\mathcal{L}$-algebra is $\mathbf{q}_\omega$-compact.

Notice that B.Plotkin uses the different terminology:

- $\mathbf{q}_\omega$-compact="logically Noetherian property";
- "equationally Noetherian"="geometrically Noetherian".

Main question of UAG

Problem

When a given algebraic structure (semigroup) is equationally Noetherian or $\mathbf{q}_\omega$-compact?

There are many examples of semigroups from such classes. For example, the famous bicyclic semigroup $B = \langle a, b \mid ab = 1 \rangle$ is not equationally Noetherian (open problem: is B $\mathbf{q}_\omega$-compact?).

Constants

Let S be a fixed semigroup. One can define the language $\mathcal{L}(S) = \{\cdot\} \cup \{s \mid s \in S\}$. $\mathcal{L}(S)$-terms: $x_1 s_1 x_2 s_2$, $s_1 x_1 x_2 s_2$....
Similarly, one can define equations, algebraic sets etc. for equations with constants.

For any semigroup S we have two different notions:

1. S is equationally Noetherian in the language $\mathcal{L}$ (i.e. any system of constant-free equations is equivalent to its finite subsystem);
2. S is equationally Noetherian in the language $\mathcal{L}(S)$ (i.e. any system with constants is equivalent to its finite subsystem).

The similar dichotomy we have for $\mathbf{q}_\omega$-compactness:

1. S is $\mathbf{q}_\omega$-compact in the language $\mathcal{L}$;
2. S is $\mathbf{q}_\omega$-compact in the language $\mathcal{L}(S)$.

Facts [see DMR]

1. (S is equationally Noetherian in the language $\mathcal{L}(S)$)$\Rightarrow$(S is equationally Noetherian in the language $\mathcal{L}$);
2. Let S be finitely generated. (S is equationally Noetherian in the language $\mathcal{L}(S)$)$\Leftarrow$(S is equationally Noetherian in the language $\mathcal{L}$);
3. (S is $\mathbf{q}_\omega$-compact in the language $\mathcal{L}(S)$)$\Rightarrow$(S is $\mathbf{q}_\omega$-compact in the language $\mathcal{L}$);

Why $B = \langle a, b \mid ab = 1 \rangle$ is not equationally Noetherian in $\mathcal{L}$? According to the fact from the last slide, in is sufficient to prove that B is not equationally Noetherian in $\mathcal{L}(S)$. Indeed,

$$\begin{cases} xba = x, \\ xb^2a^2 = x, \\ \dots \\ xb^na^n = x, \\ \dots \end{cases}$$

For example, the element ba is the solution of the 1st equation: $baba = b1a = ba$, but does not satisfy the second equation: $bab^2a^2 = b^2a^2 \neq ba$. More generally, the element b^na^n satisfies the first n equations, but does not satisfy the $(n+1)$-th equation.
Thus, B is not equationally Noetherian in $\mathcal{L}(S)$.

Why B is not equationally Noetherian in $\mathcal{L}$? We may replece constants to variables (B is f.g.)

$$\begin{cases} xba = x, \\ xb^2a^2 = x, \\ \dots \\ xb^na^n = x, \\ \dots \end{cases} \Rightarrow \begin{cases} xyz = x, \\ xy^2z^2 = x, \\ \dots \\ xy^nz^n = x, \\ \dots \end{cases}$$

Plotkin's problem for wreath products

Problem

When the wreath product of two groups A,B is

1. equationally Noetherian;
2. $\mathbf{q}_\omega$-compact, but not equationally Noetherian.
3. not $\mathbf{q}_\omega$-compact.

B. I. Plotkin "Problems in algebra inspired by universal algebraic geometry", J. Math. Sci., 139:4 (2006), 6780–6791
We study this problem for semigroups. The wreath products of semigroups is an important object which plays a central role in Krohn-Rhodes theory of finite semigroups.

Wreath products for semigroups

Let A, B be semigroups. The direct power $A^B = \prod_{b \in B} A$ of A with the index set B is the set of all tuples

$$(a_b \mid b \in B),\ a_b \in A$$

indexed by elements of B. The semigroup A^B admits the coordinate-wise multiplication

$$(a_b \mid b \in B) \cdot (a'_b \mid b \in B) = (a_b a'_b \mid b \in B).$$

Let us give the definition of the wreath product $A \wr B$ of two semigroups A, B. Since in our studies the second semigroup B is always commutative, we treat B below as a semigroup of the additive language $\mathcal{L}_+ = \{+\}$.
The *wreath product* $C = A \wr B$ of two semigroups A, B is a set of all pairs

$$\{(\mathbf{a}, b) \mid \mathbf{a} \in A^B, b \in B\}.$$

The multiplication in $A \wr B$ is defined as follows. Let $\mathbf{a} = (a_b \mid b \in B) \in A^B$, $\mathbf{a}' = (a'_b \mid b \in B) \in A^B$, then

$$(\mathbf{a}, b_1)(\mathbf{a}', b_2) = (\mathbf{a}'', b_1 + b_2), \tag{1}$$

where $\mathbf{a}'' = (a''_b \mid b \in B)$, $a''_b = a_b a'_{b+b_1}$.

Examples of multiplication in wreath products

Let $C = A \wr B$; A has a zero, and $B = \{1, 2, 3, \ldots\}$ is the additive cyclic semigroup.
We have

$$[(a_1, 0, 0, \ldots), 1][(0, a_2, 0, 0, \ldots), 1] = [(a_1 a_2, 0, 0, \ldots), 2].$$

In particular, the product

$$[(a_1, 0, 0, \ldots), n][(\underbrace{0, \ldots, 0}_{m}, a_2, 0, 0, \ldots), 1]$$

equals $[(0, 0, 0, \ldots), n + 1]$ if $n \neq m$.

Solving the equation $xy = yx$ over $C = A \wr B$, $B = \{1, 2, 3, \dots\}$

Any variable is actually a pair $x = [(x_1, x_2, \dots), \chi]$, $y = [(y_1, y_2, \dots), \omega]$. Any equation over wreath product is splitted into two expressions:

$$(x_1 y_{\chi+1}, x_2 y_{\chi+2}, \dots) = (y_1 x_{\omega+1}, y_2 x_{\omega+2}, \dots)$$

$$\chi + \omega = \omega + \chi.$$

The second equation is trivial. Thus, we obtain the infinite number is equations over A:

$$\begin{cases} x_1 y_{\chi+1} = y_1 x_{\omega+1}, \\ x_2 y_{\chi+2} = y_2 x_{\omega+2}, \\ \dots \end{cases}$$

with variable indexes χ, ω

Shift operator

The multiplication in wreath products for the cyclic semigroup B has the following sense. An element $b \in B$ defines a shift map $\sigma_b \colon A^B \to \prod A^B$,

$$\sigma_b((a_0, a_1, \dots)) = (a_b, a_{b+1}, \dots).$$

Then any product may be written as

$$(\mathbf{a}_1, b_1)(\mathbf{a}_2, b_2) = (\mathbf{a}_1 \cdot \sigma_{b_1}(\mathbf{a}_2), b_1 + b_2),$$

where $\cdot$ is the coordinate-wise product of two vectors.
E.g. an $\mathcal{L}$-equation $xy = yx$ over $A \wr B$ is equivalent to the pair of equations:

$$x\sigma_\chi(y) = y\sigma_\omega(x),\ \chi + \omega = \omega + \chi,$$

where the solutions of the first equation are considered over the direct power A^B.

We use this fact in our proofs

Let $C = A \wr B$; A has a zero, and $B = \{1, 2, 3, \ldots\}$ is the additive cyclic semigroup. The elements

$$\mathbf{a}_1 = [(\ldots), n_1], \mathbf{a}_2 = [(\ldots), n_2], \ldots, \mathbf{a}_k = [(\ldots), n_k]$$

has zero entries for all indexes $> (k-1)$. Then $\mathbf{a}_1 \mathbf{a}_2, \ldots, \mathbf{a}_k = [(0, 0, 0, \ldots), n_1 + n_2 + \ldots + n_k]$.

Proof

We have $n_1 + n_2 + \ldots + n_k > k$ in B, so all nonzero entries in $\mathbf{a}_k$ are annihilated by the shifts.

Wreath products in groups. Results

If A, B are groups, then $C = A \wr B$ is a group, and we already have several results for equations over wreath products in groups. Groups are considered in the language $\mathcal{L}_{gr} = \{\cdot, ^{-1}, 1\}$.

Result (G.Baumslag, A.Miasnikov, V.Roman'kov)

If A is non-abelian and B is infinite then $C = A \wr B$ is not equationally Noetherian in $\mathcal{L}_{gr}(C)$. Reason: an infinite chain of centralizers in C.

Theorem A

Suppose a semigroup A contains zero and a semigroup B is infinite cyclic. The wreath product $C = A \wr B$ is equationally Noetherian iff A is nilpotent (i.e. there exists a number n such that any product of n elements equals $0 \in A$).

The "if" part is obvious. If C is nilpotent any sufficiently long $\mathcal{L}$-term $t(X)$ equals $t(P) = [(0, 0, 0, \ldots), *]$ for every $P \in C^n$. Thus, equations with long parts are degenerated into a true identities.

Idea of the proof

Let us show the proof of the "only if" part.
Consider an infinite system of constant-free equations

$$\mathbf{S} = \begin{cases} x_1x_3 = x_4x_6, \\ x_1x_2x_3 = x_4x_5x_6, \\ x_1x_2^2x_3 = x_4x_5^2x_6, \\ x_1x_2^3x_3 = x_4x_5^3x_6, \\ \dots \end{cases}$$

and $\mathbf{S}_n$ be the first n equations of $\mathbf{S}$.

Since A is not nilpotent, there exist elements $a_1, a_2, \ldots, a_{n+1} \in A$ with $a_1 a_2 \ldots a_{n+1} \neq 0$. Define a point $\mathbf{P} = (\mathbf{p}_1, \ldots, \mathbf{p}_6) \in \Pi^6$ as follows:

$$\begin{aligned} \mathbf{p}_1 &= (a_1, 0, 0, \ldots), \\ \mathbf{p}_2 &= (a_1, a_2, \ldots, a_n), \\ \mathbf{p}_3 &= (\underbrace{0, \ldots, 0}_{n+1 \text{ times}}, a_{n+1}, 0, 0, \ldots), \\ \mathbf{p}_4 = \mathbf{p}_5 = \mathbf{p}_6 &= (0, 0, \ldots). \end{aligned}$$

Denote $P = (1, 1, 1, 1, 1, 1) \in B^6$, $\mathbf{0} = (0, 0, \ldots)$.
One can directly prove that all equations of $\mathbf{S}_n$ are satisfied by the given point.

Let us take the $(n+1)$-th equation of S_A:

$$x_1 x_2^{n+1} x_3 = x_4 x_5^{n+1} x_6$$

For this equation the point $(\mathbf{P}, P)$ gives

$$\begin{aligned}
\mathbf{p}_1 \sigma_1(\mathbf{p}_2) \sigma_2(\mathbf{p}_2) \ldots \sigma_n(\mathbf{p}_2) \sigma_{n+1}(\mathbf{p}_3) \quad &= \quad (a_1, 0, 0, \ldots) \\
& \qquad (a_2, a_3, \ldots, a_n, 0, 0, \ldots) \\
& \qquad (a_3, a_4, \ldots, a_n, 0, 0, \ldots) \\
& \qquad \ldots \\
& \qquad (a_n, 0, 0, \ldots) \\
& \qquad (a_{n+1}, 0, 0, \ldots) \\
&= \quad (a_1 a_2 a_3 \ldots a_{n+1}, 0, 0, \ldots) \neq \mathbf{0}.
\end{aligned}$$

Thus, the system $\mathbf{S}$ is not equivalent to its finite subsystems.

Theorem B.

The wreath product $C = A \wr B$ of two semigroups A, B is $\mathbf{q}_\omega$-compact if A contains a zero and B is infinite cyclic.

Interesting property: a semigroup A may be not $\mathbf{q}_\omega$-compact, but the whole wreath product $C = A \wr B$ is $\mathbf{q}_\omega$-compact.