Tame and wild automorphisms of free algebras

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Automorphisms of free groups

Let F_n be a free group of rank n with a set of free generators $X = \{x_1, \ldots, x_n\}$.

Elementary Nielsen automorphisms:
(1) n_i sends x_i to x_i^{-1} and leaves the other elements of X unchanged;
(2) n_{ij} sends x_i to $x_i x_j$ $(i \neq j)$ and leaves the other elements of X unchanged.

Jakob Nielsen, 1924 (a) The automorphism group $\mathrm{Aut} F_n$ of F_n is generated by all elementary Nielsen automorphisms.
(b) The automorphism group $\mathrm{Aut} F_n$ is a finitely presented group.

Nielsen, 1921 – Schreier, 1927 Subgroups of free groups are free.

Automorphisms of $K[x, y]$

Let K be a field and $K[x, y]$ be the polynomial algebra over K in two variables x, y. Automorphisms of $K[x, y]$ of the form

$$\begin{array}{rcl} x & \mapsto & \alpha x + g(y) \\ y & \mapsto & y \end{array}$$

and

$$\begin{array}{rcl} x & \mapsto & x \\ y & \mapsto & \beta y + f(x) \end{array}$$

are called **elementary**.

Jung, 1942, van der Kulk, 1953 Every automorphism of $K[x, y]$ is a product of elementary automorphisms.

Amalgamated free product structure

van der Kulk, 1953, Shafarevich, 1966

The group of automorphisms of $K[x,y]$ admits the following amalgamated free product structure:

$$\mathrm{Aut}(K[x,y]) \cong \mathrm{Af}_2(K) *_H \mathrm{Tr}_2(K),$$

where

$$\mathrm{Af}_2(K) = \left\{ \begin{array}{ccc} x & \mapsto & \alpha_1 x + \beta_1 y + \gamma_1 \\ y & \mapsto & \alpha_2 x + \beta_2 y + \gamma_2 \end{array} \right\}$$

is the group of affine automorphisms,

$$\mathrm{Tr}_2(K) = \left\{ \begin{array}{ccc} x & \mapsto & \alpha x + g(y) \\ y & \mapsto & \beta y + \gamma \end{array} \right\}$$

is the group of triangular automorphisms, and $H = \mathrm{Af}_2(K) \cap \mathrm{Tr}_2(K)$.

Elementary automorphisms

Let $A = K[x_1, x_2, \ldots, x_n]$ be a polynomial algebra. An automorphism ϕ of A such that

$$x_i \mapsto \alpha x_i + f,$$
$$x_j \mapsto x_j, j \neq i,$$

where $0 \neq \alpha \in K,\ f \in K[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, is called **elementary**.

An automorphism is called **tame** if it is a product of elementary automorphisms. A nontame automorphism is called **wild**.

The subgroup $\mathrm{TA}(A)$ of $\mathrm{Aut}(A)$ generated by all elementary automorphisms is called the subgroup of **tame** automorphisms.

The Nagata automorphism

Masayoshi Nagata, 1972 Automorphism σ of $K[x, y, z]$:

$$\begin{aligned} \sigma(x) &= x + 2(y^2 - xz)y + (y^2 - xz)^2 z, \\ \sigma(y) &= y + (y^2 - xz)z, \\ \sigma(z) &= z. \end{aligned}$$

Nagata Automorphism σ is a non-tame automorphism of $K[z][x, y]$.

Martha Smith, 1989 The Nagata automorphism is stably tame.

Shestakov–U., 2001 The Nagata automorphism is wild $(\mathrm{char}(K) = 0)$.

Free associative algebras

Makar-Limanov, 1970, Czerniakiewicz, 1971-1972 Every automorphism of the free associative algebra $K\langle x, y\rangle$ is tame and

$$\mathrm{Aut}(K[x,y]) \cong \mathrm{Aut}(K\langle x,y\rangle) \cong \mathrm{Af}_2(K) *_H \mathrm{Tr}_2(K).$$

David Anick Let $K\langle x, y, z\rangle$ be a free associative algebra in three variables. Consider the automorphism δ defined by

$$\begin{aligned} \delta(x) &= x + z(xz - zy), \\ \delta(y) &= y + (xz - zy)z, \\ \delta(z) &= z. \end{aligned}$$

U., 2004 The Anick automorphism is wild $(\mathrm{char}(K) = 0)$.

Poisson algebras

An associative and commutative algebra $\langle P, \cdot \rangle$ over a field K endowed with a bracket $\{x, y\}$ (a Poisson bracket) is called **a Poisson algebra** if
(1) P is a Lie algebra under $\{x, y\}$,
(2) P satisfies the the Leibniz identity:

$$\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}.$$

The **symplectic Poisson** algebra P_n of index $n \geq 1$ is the polynomial algebra in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$ endowed with the Poisson bracket defined by

$$\{y_i, x_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where $1 \leq i, j \leq n$.

Poisson enveloping algebras

Let L be an arbitrary algebra with a skew-symmetric bilinear operation $[\cdot, \cdot]$ over a field K of characteristic 0 and let

$$e_1, e_2 \ldots, e_n, \ldots$$

be a linear basis of L.

Then there exists a unique bracket $\{\cdot, \cdot\}$ satisfying the Leibniz identity on the polynomial algebra

$$K[e_1, e_2 \ldots, e_n, \ldots]$$

such that

$$\{e_i, e_j\} = [e_i, e_j].$$

The algebra $\langle K[e_1, \ldots,], \cdot, \{\cdot, \cdot\}\rangle$ will be called the **Poisson enveloping algebra** of L and will be denoted by $P(L)$.

If L is a free Lie algebra then $P(L)$ is a free Poisson algebra.

Automorphisms of Poisson algebras

Makar-Limanov, Turusbekova, U., 2009 Every automorphism of a free Poisson algebra $P\langle x, y\rangle$ in two variables over a field of characteristic zero is tame and

$$\mathrm{Aut}(K[x,y]) \cong \mathrm{Aut}(K\langle x,y\rangle) \cong \mathrm{Aut}(P\langle x,y\rangle) \cong \mathrm{Af}_2(K) *_H \mathrm{Tr}_2(K).$$

Makar-Limanov, U., 2011 Let $P(x,y) = Q(P\langle x,y\rangle)$ be the free Poisson field in two variables x, y over a field K of characteristic 0. Then

$$\mathrm{Aut}\, P(x,y) \cong \mathrm{Aut}\, K(x,y) = Cr_2(K).$$

The Nagata automorphism is a wild automorphism of free Poisson algebras in three variables.
I. Shestakov, 2005 An analogue of the Anick automorphism is also a wild automorphism of free Poisson algebras in three variables.

Candidates to be nontame in dimension 4

Nagata–Anick Consider the following automorphism of $K[x, y, z, t]$ or $K\langle x, y, z, t\rangle$

$$\begin{aligned} x &\longmapsto x + t(xz - ty), \\ y &\longmapsto y + (xz - ty)z, \\ z &\longmapsto z, \\ t &\longmapsto t. \end{aligned}$$

Martha Smith, 1989 The Nagata and the Anick automorphisms are stably tame.

Approximately tame automorphisms

Let

$$A = A_0 \oplus A_1 \oplus \ldots \oplus A_k \oplus \ldots$$

be the standard degree grading of the polynomial algebra $A = K[x_1, x_2, \ldots, x_n]$, i.e., A_k is the linear span of monomials of degree k.

Let $\mathrm{IA}_i(A)$ be the subgroup of all automorphisms of A that induces the identity automorphism on the factor-algebra $A/(A_i + A_{i+1} \ldots)$. We have

$$\mathrm{Aut}(A) = \mathrm{IA}_0(A) \supset \mathrm{IA}_1(A) \supset \ldots \supset \mathrm{IA}_k(A) \supset \ldots.$$

An automorphism ϕ is called **approximately tame** if there exists a sequence of tame automorphisms $\{\psi_k\}_{k\geq 0}$ such that $\phi\psi^{-1} \in \mathrm{IA}_k(A)$. (limit in the formal power series topology)

Shafarevich–Anick

Shafarevich, 1981 – Anick, 1983 Every automorphism of $A = K[x_1, x_2, \dots, x_n]$ is approximately tame.

Corollary The Nagata automorphism is approximately tame.

Problem 1 Is every automorphism of a free associative algebra approximately tame?

Problem 2 Is every automorphism of a free Poisson algebra approximately tame?

Nielsen-Schreier varieties of algebras

A variety of universal algebras is called **Nielsen-Schreier**, if any subalgebra of a free algebra of this variety is free, i.e., an analog of the classical Nielsen-Schreier Theorem holds.

A.G. Kurosh, 1947 The variety of all nonassociative algebras is Nielsen-Schreier.
A.I. Shirshov, 1954 The varieties of all commutative and anti-commutative algebras are Nielsen-Schreier.
A.I. Shirshov, 1953, E. Witt, 1956 The varieties of all Lie and p-Lie algebras are Nielsen-Schreier.
A.A. Mikhalev, 1985,1988, Stern, 1986 The varieties of all Lie superalgebras, color Lie superalgebras, p-Lie superalgebras are Nielsen-Schreier.

V. Dotsenko, U., 2022

(1) The variety of pre-Lie (also known as right-symmetric) algebras,
(2) the variety of Lie-admissible algebras,
(3) the variety of nonassociative algebras satisfying the identity

$$xx^2 + \alpha x^2 x = 0,$$

for every given $\alpha \neq 1$,
(4) the variety of nonassociative algebras satisfying the identity

$$x(x(\cdots(xx^2))) = 0.$$

Automorphisms of free algebras

P. Cohn, 1964 Every automorphism of a finitely generated free Lie algebra is tame.

J. Lewin, 1968 Every automorphism of a finitely generated free algebra of a Nielsen-Schreier variety of algebras is tame.

U., 2004 Let A be a a finitely generated free algebra of a Nielsen-Schreier variety of algebras. The defining relations of the group of automorphisms $\mathrm{Aut}(A)$ with respect to the set of all elementary automorphisms are described.

Free Lie algebras of rank 3

Alimbaev, Nauryzbaev, U., 2020 The group of automorphisms of a free Lie algebra $L_3 = \mathrm{Lie}\langle x, y, z\rangle$ admits an amalgamated free product structure

$$\mathrm{Aut}(L_3) = \mathrm{GL}_3(K) *_H T_3,$$

where T_3 is the subgroup of all automorphisms of the form

$$\begin{array}{ccc} x & \mapsto & \alpha x + f \\ y & \mapsto & \beta_2 y + \beta_3 z \\ z & \mapsto & \gamma_2 y + \gamma_3 z \end{array}$$

where $\alpha, \beta_i, \gamma_i \in K$, $f \in \mathrm{Lie}\langle y, z\rangle$, and $H = \mathrm{GL}_3(K) \cap T_3$.

M.A. Shevelin, 2012 Every finite group of automorphisms of a free Lie algebra in three variables over a field of characteristic zero is linearizable.

Question Is there an analogue of this result for groups?

Automorphisms of free metabelian groups

Let F_n be the free group of rank n. Then $M_n = F_n/F_n''$ is the free metabelian group of rank n.

Bachmuth, 1965 $\mathrm{IAut}(M_2) = \mathrm{Inn}(M_2)$. Hence every automorphism of M_2 is tame.

Chein, 1968 M_3 has wild automorphisms.

Bachmuth, Mochizuki, 1967 $\mathrm{Aut}(M_3)$ is not even finitely generated.

Roman'kov, 1992 M_3 has wild primitive elements.

Bachmuth, Mochizuki, 1985–Roman'kov, 1985 Every automorphism of M_n for $n \geq 4$ is tame.

Let L_n be the free Lie algebra of rank n. Then $M_n = L_n/L_n''$ is the free metabelian Lie algebra of rank n.

Chein or one-row automorphisms $\phi = (f, x_2, \ldots, x_n)$, i.e.,

$$x_1 \mapsto f,$$
$$x_i \mapsto x_i, i \geq 2.$$

Example M_n, $n \geq 3$

$$x_1 \mapsto x_1 + [[x_2, x_3], x_1],$$
$$x_i \mapsto x_i, i \geq 2.$$

Exponential or inner automorphisms Let $a \in M_n'$. Then $\mathrm{ad}(a)^2 = 0$ and

$$\exp(\mathrm{ad}(a)) = (x_1 + [a, x_1], x_2 + [a, x_2], \ldots, x_n + [a, x_n])$$

is an automorphism

Automorphisms of free metabelian Lie algebras II

Artamonov, 1977 Every nontrivial inner automorphism of M_2 is wild.

An automorphism is called **absolutely wild** if it is not approximately tame.

Bryant, Drensky, 1993 M_2 and M_3 have absolutely wild automorphisms.

An automorphism of M_n is called **strictly wild** if it does not belong to the subgroup generated by all tame, one-row, and inner automorphisms.

Roman'kov, 2008 M_3 has strictly wild automorphisms.

Kabanov, Roman'kov, 2009 M_3 has strictly wild primitive elements.

Automorphisms of free polynilpotent Lie algebras

Bryant, Drensky, 1993 Every automorphism of M_n for $n \geq 4$ is approximately tame.

$\mathfrak{N}_c$ is the variety of all nilpotent Lie algebras of class $\leq c$
$\mathfrak{A}$ is the variety of abelian Lie algebras
A variety of Lie algebras of the form $\mathfrak{N}_{c_1}\mathfrak{N}_{c_2}\ldots\mathfrak{N}_{c_k}$ is called polynilpotent.

Drensky, 1992, Papistas, 1993, Bahturin, Shpilrain, 1995 Let $\mathfrak{M}$ be a polynilpotent variety of Lie algebras that is not $\mathfrak{A}^2$. Then every free algebra of $\mathfrak{M}$ of rank $n \geq 2$ has wild automorphisms.

V. Dotsenko, U., 2023 In fact, these wild automorphisms are absolutely wild.

Automorphisms of free metabelian Lie algebras III

Bahturin, Nabiev, 1992 Every nontrivial inner automorphism of M_n is wild.

Özkurt, Ekici, 2008 The automorphism ψ defined by

$$x_1 \mapsto x_1 + [[x_1, [x_2, x_3]], x_4], \;\; x_i \mapsto x_i, \;\; i \geq 2$$

is wild.

Both papers contain elementary mistakes.

Problem Is it true that every automorphism of the free metabelian Lie algebra M_n of rank $n \geq 4$ tame?